An Application of Bridgeland Stability to the Geometry of Abelian Surfaces

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Abstract

A key property of projective varieties is the very ampleness of line bundles as this provides embeddings into projective space and allows us to express the variety in equational terms. In this thesis we study the general version of this property which is k- very ampleness of line bundles. We introduce the notation of critical k-very ampleness and compute it for abelian surfaces. The property of k-very ampleness is usually discussed using tools from divisor theory but we take a different approach and use methods from derived algebraic geometry as part of program to use properties of the derived category of a variety to access the geometry of the variety. In particular, we use the Fourier-Mukai transform, moduli spaces of sheaves and properties of Bridgeland stability. We compute walls for certain Bridgeland stable spaces and certain Chern characters and to complete the picture we study the moduli spaces of torsion sheaves with minimal first Chern class and we go on to compute the walls for these as well building on tools developed earlier in the thesis.

Lay Summary

Algebraic geometry is a branch of geometry studying objects called algebraic varieties which are solution sets of polynomials in space. In recent years, the new way of studying varieties and geometrical problems is by using very abstract methods. We are interested in the particular case of a complex abelian surface which is a two dimensional "complex torus". Sheaf theory is playing a vital role in algebraic geometry and sheaves are objects that we use to give data about algebraic variety. A sheaf provides a continuously varying set on a variety. One way of using sheaves is to observe that they come in families called moduli spaces and sometimes we use the geometry of the moduli spaces to understand the space of the original variety. When we talk about spaces, the line bundles, or more generally vector bundles, associated to these spaces became vital things to study the geometry of the spaces.

The notion of k-very ampleness for line bundles was introduced in the eighties. We introduce the notation of critical k-very ampleness which is an integer and we compute it for abelian surfaces. One of the main technical tools that we use in our study is the Fourier-Mukai transforms. These are sheaf versions of the classical Fourier transform used in analysis. We also use the notion of stability that was introduced by Bridgeland. Stable sheaves form better behaved moduli but the usual notion of stability is not well preserved under the Fourier-Mukai transforms. Bridgeland stability plugs the gap and we study how the transforms interact with Bridgeland stability and the k-very ampleness of line bundles. We give an example of how to apply this to understand something of the geometry of an abelian surface.

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To my family.

Contents

Chapter 1

Introduction

In this thesis, we will be concerned with a special type of algebraic variety: the abelian variety and especially abelian surfaces over the complex numbers. In particular, we will study algebraic sheaves on such surfaces. First, we will review some basic definitions and examples for algebraic varieties and coherent sheaves from the standard references (see [Har77] and [Mum88]).

1.1 Varieties

Let k be a fixed algebraically closed field. We define **affine n-space** $\mathbb{A}^n$ over k to be the set of all n-tuples of elements of k. An element P in $\mathbb{A}^n$ is called a point with coordinate $P = (a_1, \ldots, a_n)$ and $a_i \in k$ for $i = 1, \ldots, n$. An irreducible closed subset of $\mathbb{A}^n$ is called an **affine variety**.

Example 1.1.1. *A **projective n-space** over k is a set of $(n+1)$- tuples $(a_0, \ldots, a_n)$ of elements of k such that some $x_i \neq 0$ modulo the equivalence relation*

$$(a_0, \ldots, a_n) \sim (\alpha a_0, \ldots, \alpha a_n)$$

for all $\alpha \in k$ and $\alpha \neq 0$. We denote it by $\mathbb{P}_k^n$ or simply $\mathbb{P}^n$.

In other words, $\mathbb{P}^n$ is the quotient of the set $\mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\}$ with the relation $P_1 \sim P_2$

if P_1 and P_2 are lying on the same line through the origin.

Let $k[x_0,...,x_n]$ (or simply $k[x]$) be the polynomial ring in $n+1$ variables over k and $T \subset k[x]$.

Definition 1.1.2. *We define a* ***closed algebraic set*** *Y on $\mathbb{P}^n$ to be*

$$Y = \{P \in \mathbb{A}^n | f(P) = 0, \textit{for all homogeneous} f \in T\},$$

the set of common zeros of the elements of T.

Also, we define its ***homogeneous ideal*** *$I(Y)$ as follows*

$$I(Y) \textit{ is the set } I(Y) = \{f \in k[x] | f \textit{ is homogeneous, } f(P) = 0 \textit{ for all } P \in Y\}.$$

We say that $I(Y)$ is ***prime*** *if $f, g \in k[x]$ and $fg \in I(Y)$, then $f \in I(Y)$ or $g \in I(Y)$.*

Oscar Zariski introduced a topology on algebraic varieties to allow using topology tools to solve problems in algebraic varieties. This topology is called ***Zariski topology****. We defined it on $\mathbb{A}^n$ ($\mathbb{P}^n$) to be a topology whose open sets are the complements of the algebraic sets on $\mathbb{A}^n$ ($\mathbb{P}^n$).*

Definition 1.1.3. *A* ***projective variety*** *is an irreducible algebraic set in $\mathbb{P}^n$ with the induced Zariski topology and the dimension of projective variety is the dimension of its Zariski topological space.*

Example 1.1.4. *The following are examples of projective varieties in $\mathbb{P}^n$:*

- *If n=2, solutions set of a cubic curve C defined in homogeneous coordinates by*

$$zy^2 = x^3 - xz^2$$

  is a projective variety in $\mathbb{P}^2$.

It turns out such cubic curves admit group laws.

Definition 1.1.5. *Over algebraically closed field k, we say that a variety X is a* ***complete variety*** *if for all variety Y, the projection map*

$$\pi_2 : X \times Y \to Y$$

is a closed map.

This plays the role of compactness for topological spaces for which this is an alternative definition.

Theorem 1.1.6. *([Mum88], p.77) $\mathbb{P}^n$ is a complete variety.*

Example 1.1.7. *Any projective variety is complete variety because the closed subvariety of a complete variety is complete.*

1.2 Sheaf Theory and Schemes

Let X be a topological space, a category $\mathfrak{Top}(X)$ of the open subsets of X and the morphisms are the inclusion map, i.e. if $U \not\subseteq V$ then $\mathrm{Hom}(U, V)$ is empty and there is only one element in $\mathrm{Hom}(U, V)$ if $U \subseteq V$. We define a **presheaf** $\mathscr{F}$ to be a functor from the category $\mathfrak{Aop}(X)$ to the category $\mathfrak{Ap}(X)$ of abelian groups and we call $\mathscr{F}(U)$ **set of sections of** $\mathscr{F}$ **over** U. A sheaf $\mathscr{F}$ is a presheaf satisfying extra conditions as follows

Definition 1.2.1. *If $\mathscr{F}$ is a presheaf on X, we say that $\mathscr{F}$ is a* **sheaf** *if the following conditions hold:*

1. *If U is an open set, $\{V_i\}$ is an open cover of U, and s is a section of $\mathscr{F}$ over U such that $s|_{V_i} = 0$ for all i, then $s = 0$.*
2. *If U is an open set, $\{V_i\}$ is an open cover of U, and s_i is a section of $\mathscr{F}$ over V_i for all i satisfying the property for each i, j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists a section s of $\mathscr{F}$ over U such that $s|_{V_i} = s_i$ for all i.*

A subsheaf of a sheaf $\mathscr{F}$ is a sheaf $\mathscr{F}'$ such that for any open subset U of X, the group $\mathscr{F}(U)$ is a subgroup of $\mathscr{F}'(U)$ and the restriction maps of the sheaf $\mathscr{F}'$ are induced by those of $\mathscr{F}$. To study the behaviour of a sheaf $\mathscr{F}$ at point x, we look at a **stalk** of $\mathscr{F}$ at a point x on X, $\mathscr{F}_x$, which is the direct limit of the group $\mathscr{F}(U)$ for all open neighborhood U of x.

Example 1.2.2. *Let U be an open subset of X and $\mathcal{O}(U)$ be the ring of regular functions from U to k. Then $\mathcal{O}(U)$ is a sheaf of rings and if $V \subset U$, the morphism is a restriction map $\rho_{UV} : \mathcal{O}(U) \to \mathcal{O}(V)$. We call $\mathcal{O}$ the **sheaf of regular functions** on X.*

Definition 1.2.3. *A **sheaf of** $\mathcal{O}_X$**-modules** is a sheaf $\mathscr{F}$ on X such that*

1. *for any open subset U of X, $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module and,*
2. *for each inclusion $U \hookrightarrow V$ of open sets, the restriction homomorphism $\mathscr{F}(U) \to \mathscr{F}(V)$ is compatible with the module structures via $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.*

Let $\mathscr{F}$ be a sheaf on a projective variety X of $\mathcal{O}_X$-modules, then $\mathscr{F}$ is said to be of **coherent** if for each $x \in X$ there is an open subsets U of X such that $x \in U$ and a sequence of $\mathcal{O}_X|_U$-modules of the form

$$\mathcal{O}_X^m|_U \to \mathcal{O}_X^n|_U \to \mathscr{F}|_U \to 0$$

where $m, n \in \mathbb{N}$.

Definition 1.2.4. *The **spectrum** of a ring A is the pair consisting of the Zariski topoloy on the space* Spec A *together with the sheaf of rings $\mathcal{O}$ where* Spec A *is the set of all prime ideals of A.*

Definition 1.2.5. *An **affine scheme** is a a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the spectrum of some ring. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that for each $x \in X$ there is an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.*

Over a closed field k, we define the **length** of a zero dimensional subscheme Z to be the dimension of the k-vector space $\mathcal{O}_X(X)$.

Example 1.2.6. *Let X be a variety over fixed field k, a zero dimensional subscheme Z of length m is a set of m points $Z = \{z_1, ..., z_m\}$ such that $z_i \neq z_j$ for all $i \neq j$.*

Example 1.2.7. *Let k be a closed field, here are some examples of 0-dimensional subschemes of $\mathbb{A}^n$*

1. *Let $Z = \operatorname{Spec} k[x]/(x^2)$. Then Z is a 0-dimensional subschemes of $\mathbb{A}^1$ of length two.*

2. *Let $Z = \operatorname{Spec} k[x,y]/(x^2, y^2, xy)$. Then Z is a 0-dimensional subschemes of $\mathbb{A}^2$ of length three.*

Definition 1.2.8. *Suppose Y is a set of n distinct points $y_1, \ldots, y_n \in \mathbb{P}^2$ whose corresponding ideals are $I_1, \ldots, I_n$, and $m_1, \ldots, m_n$ are positive integers. The subscheme of $\mathbb{P}^2$ defined by the ideal*

$$I = I_1^{m_1} \cap \cdots \cap I_n^{m_n}$$

*is called the **subscheme of fat points***

$$Z = m_1 y_1 + \cdots + m_n y_n.$$

Definition 1.2.9. *We say that a sheaf $\mathscr{F}$ of dimension d is **pure** if for all non-trivial coherent subsheaf $\mathscr{E} \subset \mathscr{F}$, the dimension of $\mathscr{E}$ is d.*

Definition 1.2.10. *Let $\mathscr{F}$ be a sheaf, we define a **support of** $\mathscr{F}$, $\operatorname{Supp}(\mathscr{F})$, to be the set $\operatorname{Supp}(\mathscr{F}) = \{x \in X | \mathscr{F}_x \neq 0\}$. The maximal dimension over any subscheme is called the **dimension of** $\mathscr{F}$ and we denote it by $\dim(\mathscr{F})$. This can be given the structure of a scheme.*

Definition 1.2.11. *If X is an algebraic variety over a field k and $\mathscr{F} \in \operatorname{Coh}(X)$, we denote the Chern classes of $\mathscr{F}$ by $c_i(\mathscr{F}) \in H^{2i}(X,k)$ and the Chern character of $\mathscr{F}$ by $\operatorname{ch}(\mathscr{F}) \in \bigoplus_i^{\dim X} H^{2i}(X,k)$ such that*

- *If we have a short exact sequence*

$$0 \to \mathscr{E} \to \mathscr{G} \to \mathscr{F} \to 0,$$

then $\mathrm{ch}(\mathscr{G}) = \mathrm{ch}(\mathscr{E}) + \mathrm{ch}(\mathscr{F})$.

- *If* $\mathscr{F}, \mathscr{E} \in \mathrm{Coh}(X)$, *then* $\mathrm{ch}(\mathscr{F} \otimes \mathscr{E}) = \mathrm{ch}(\mathscr{F}) \cdot \mathrm{ch}(\mathscr{E})$.

- $c_0(\mathscr{F}) = r(\mathscr{F})$, *the rank of* $\mathscr{F}$.

Definition 1.2.12. *Let $\mathscr{F}$ be a coherent sheaf on a projective variety X over algebraically closed field k. We define the* ***Euler charactaristic*** *of $\mathscr{F}$ by*

$$\chi(\mathscr{F}) = \sum (-1)^i h^i(X, \mathscr{F}),$$

where $h^i(X,\mathscr{F}) = \dim_k H^i(X,\mathscr{F})$

Let X be a variety over fixed field k, we denote by $\mathrm{Coh}(X)$ the **category of coherent sheaves on** X. A non-zero sheaf $\mathscr{F}$ with rank $r(\mathscr{F}) = 0$ is called a **torsion sheaf**, and we call a sheaf **torsion-free** if it has no torsion subsheaf. On the other hand, we denote the dual of a coherent sheaf $\mathscr{F}$ by $\mathscr{F}^*$ and it is the coherent sheaf given by $\mathscr{F}^* = \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$.

Consider the canonical sheaf homomorphism

$$g : \mathscr{F} \to \mathscr{F}^{**}$$

from $\mathscr{F}$ into its bidual sheaf $\mathscr{F}^{**}$. The kernel of g, $\ker(g)$, is a torsion subsheaf of $\mathscr{F}$ (see [OSS], p.148). In the case of $\mathscr{F}$ is torsion-free sheaf, then g injects and there is a short exact sequence

$$0 \to \mathscr{F} \to \mathscr{F}^{**} \to Q \to 0$$

where Q is the quotient sheaf with $r(Q) = 0$.

Definition 1.2.13. *Let $\mathscr{F}$ be a coherent sheaf on a projective variety X over algebraically closed field k. We say that $\mathscr{F}$ is **generated by global sections** if $\mathscr{F} \to \mathscr{F} \otimes \mathcal{O}_x$ surjects for all $x \in X$.*

Definition 1.2.14. *We denote by $\mathrm{Coh}^{\leq d}(X)$ the full subcategory of $\mathrm{Coh}(X)$ whose objects are sheaves of dimension $\leq d$.*

Definition 1.2.15. *Let E be a coherent sheaf on X, we define the singularity set $\mathrm{Sing}(E)$ of $E \in \mathrm{Coh}(X)$ as the locus where E is not locally free,*

$$\mathrm{Sing}(E) = \{x \in X : \mathrm{Ext}^1(E, \mathcal{O}_x) \neq 0\}.$$

The following are some examples of coherent sheaves on varieties:

Example 1.2.16. *Let X be an algebraic variety with a structure sheaf $\mathcal{O}_X$. An ideal sheaf on X is a sheaf of modules $\mathcal{I}$ and it is subsheaf of $\mathcal{O}_X$. If Y is a closed subvariety of X, the kernel of the morphism $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is the ideal sheaf of Y and we denote it by $\mathcal{I}_Y$, where $i : Y \to X$ is the inclusion map.*

Example 1.2.17. *If X is a projective variety, then*

- *If $X = \mathcal{C}$ is a smooth projective curve and $Y \subset \mathcal{C}$ a 0-dimensional subscheme, then $\mathcal{O}(Y)$ is a line bundle and $\mathcal{I}_Y = \mathcal{O}(Y)^*$.*
- *In the case of $X = S$ is a smooth projective surface and $Y \subset S$ a curve then $\mathcal{I}_Y$ is the kernel of the canonical map $\mathcal{O}_S \to i_*\mathcal{O}_Y$ which surjects.*

1.3 Abelian Varieties and Abelian Surfaces

In this section we will review some standard material principally from [Mum74] and [BL04].

Definition 1.3.1. *We say that X is an **abelian variety** if X is a complete algebraic variety over an algebraically closed field k with the following maps (morphisms of varieties)*

- *a group law morphism* $m : X \times X \to X$ *which takes* (x_1, x_2) *to* $x_1 + x_2$ *and,*

- *the inverse map,* $(-1) : X \to X$ *are morphisms of varieties.*

We denote the zero element in X by e.

A **complex torus** of dimension g is a quotient $X = \mathbb{C}^g / \Lambda$ where Λ is a lattice in $\mathbb{C}^g$. In the case $k = \mathbb{C}$, an abelian variety is a complex torus with a positive definite line bundle L.

Example 1.3.2. *If* $g = 1$*, let* $X = \mathbb{C}/\Lambda$ *where* Λ *is a lattice in* $\mathbb{C}$ *generated by linearly independent elements* $x_1, x_2 \in \mathbb{C}$*. We call a one dimensional abelian variety an* ***elliptic curve****.*

In two dimensions we call them abelian surfaces. We will mostly be interested in abelian surfaces. The group of line bundles on X is denoted by $\mathrm{Pic}(X)$ the subgroup $\mathrm{Pic}^0(X)$ is the group of line bundles L on X such that $c_1(L) = 0$. The group $\mathrm{Pic}^0(X)$ is naturally isomorphic to an abelian variety $\hat{X}$ called **the dual abelian variety of** X with the same dimension. There is a short exact sequence of groups

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0$$

where $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ and it is called **Néron-Severi group**. The rank of the Néron-Severi group $NS(X)$ is called the **Picard number**, $\rho(X)$, of X.

Example 1.3.3. *Let* X *be an abelian variety if* X *is a complete algebraic variety of dimension one over an algebraically closed field* k *and* L *a line bundle on* X *such that* $L = \mathcal{O}(e)$*. Then the map*

$$f : X \to \hat{X} \quad \textit{given by} \quad x \mapsto L_x \otimes L^*$$

is an isomorphism and the inverse f^{-1} *is defined as* $\mathcal{P}_{\hat{x}} \mapsto \mathrm{Supp}(\mathcal{O}_X / L^* \otimes \mathcal{P}^*_{\hat{x}})$.

For an abelian variety, the tangent bundle TX is trivial so $K_X = \mathcal{O}_X$ and Serre duality takes the simple form

$$\mathrm{Ext}^i(A,B) \cong \mathrm{Ext}^{g-i}(B,A)^*$$

for any sheaves $A, B \in \mathrm{Coh}(X)$ (and so this also holds for objects in $\mathcal{D}(X)$), where $g = \dim(X)$

For us, a **polarization on an abelian variety** X is a choice of ample line bundle L. This polarization defines the cononical homomorphism

$$\varphi_L : X \to \hat{X},\ \varphi_L(x) = \tau_x^* L \otimes L^*$$

where $\tau_x : X \to X$ is a translation map takes $a \in X$ to $a + x$. We denote the kernel of φ_L by $K(L)$ and it has structure of an abelian subvariety.

Definition 1.3.4. *Let X ba an abelian variety. We define a **Poincaré bundle** $\mathcal{P}$ on $X \times \hat{X}$ to be a holomorphic line bundle on $X \times \hat{X}$ satisfying*

i. *$\mathcal{P}|_{X\times\{L\}} \sim L$ for all $L \in Pic^0(X)$, and*

ii. *$\mathcal{P}|_{\{0\}\times\hat{X}}$ is trivial.*

Example 1.3.5. *([HP05], p.424) Let X be an abelian variety with $\dim X = 1$, $X = \hat{X}$ and then*

$$\mathcal{P} = \mathcal{O}(\Delta - X \times \{e\} - \{e\} \times X),$$

and $\Delta \subset X \times X$ is the diagonal.

Let L be a line bundle on a complex torus $X = V/\Lambda$ of dimension g. We define the degree an the type of the polarization L as follows

Definition 1.3.6. *[BL04] Let L with $c_1(L) = H$ be a line bundle on X and $E =$Im H is integer-valued on Λ, then there is a basis $\lambda_1, ..., \lambda_g, \mu_1, ..., \mu_g$ of Λ with respect to which*

E is given by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = \mathrm{diag}(d_1, \ldots, d_g)$ *where* $d_i \in \mathbb{Z}^+$ *and* $d_i | d_{i+1}$ *for* $i = 1, ..., g-1$. *The vector* $(d_1, \ldots, d_g)$ *is the* ***type of the polarization*** L *, and the* ***degree of the polarization*** L *is the product*

$$\deg(L) = d_1 \cdot ... \cdot d_g = \frac{1}{g!} c_1(L)^g.$$

Note that the divisors $d_1, \ldots, d_g$ are uniquely determined by E, Λ and L.

If L is a polarization on an abelian variety X, we say that L is **principal** if

$$\deg(\phi_L) = \frac{c_1(L)^g}{g!} = 1,$$

and the pair (X, L) is called **polarized abelian variety**.

Definition 1.3.7. *Let X be an abelian variety with* $\dim(X) = g$ *and L a line bundle on X. The rational map $\phi_L : X \to \mathbb{P}_N$ is a meromorphic map associated to L and defined as the following: If $a_0, \ldots, a_N$ is a basis of $H^0(L)$, then $\phi_L(x) = (a_0(x), \ldots, a_N(x))$ where not all $a_i(x)$ are zero.*[1]

We say that a line bundle $L \in \mathrm{Pic}(X)$ is **very ample** if ϕ_L is embedding and it is **ample** if there is $n \geq 1$ such that L^n is very ample, i.e. ϕ_{L^n} is embedding. The following proposition is a characterization for L to be ample:

Proposition 1.3.8. *[BL04, Proposition 4.5.2] If L is a line bundle on an abelian variety X, then the following are equivalent.*

1. *L is an ample line bundle.*

2. *$H^0(L) \neq 0$ and $K(L)$ is finite.*

In this thesis we are mainly interested in studying a generalization of the very-ampleness property called k-very ampleness of line bundles. We study it in detail for

[1] We will not use the notation ϕ_L in what follows; redefine ϕ_L to be critical k-very ampleness.

an abelian surface in chapters 3 and 4. The following section will give some historical information about this property.

1.4 Theoretical and Historical Background

In 1988, the first notion of k-very ampleness was introduced by Beltrametti and Sommese in [BS91]. This was to understand the idea of higher order embeddings and to generalize the notations of very ample and spannedeness. According to their definition, an ample line bundle L on a variety V is **k-very ample** if for each 0-dimensional subscheme $X \subset V$ of length $k+1$, the restriction map $H^0(V,L) \to H^0(\mathcal{O}_X)$ is surjective.

As a result of this paper, Catanese and Göettsche in [CG90] gave another characterization of k-very ample, the map

$$\varphi_k : \mathrm{Hilb}^{k+1} V \to \mathrm{Grass}(k+1, \Gamma(L))$$

to the Grassmannian of $k+1$-dimensional linear subspaces of $\Gamma(L)$ is an embedding if and only if L is k-very ample.

On the other hand, we say that an ample line bundle L on S is k**-spanned** if and only if for each distinct set of points $z_1, \dots, z_r$ and for any positive integers $k_1, \dots, k_r$ such that $\sum_{i=1}^{r} k_i = k+1$ the map

$$\Gamma(L) \to \Gamma(L \times \mathcal{O}_Z)$$

is surjective, where $\mathcal{O}_Z$ is a 0-dimensional subscheme defined by the ideal sheaf $\mathcal{I}_Z$ where $\mathcal{I}_Z \mathcal{O}_{S,z}$ is $\mathcal{O}_{S,z}$, for $z \notin \{z_l, ..., z_r\}$ and $\mathcal{O}_{S,z_i}$, is generated by (x_i, y_i) at z_i, with (x_i, y_i) local coordinates at z_i on S, $i = 1, ..., r$.

Many authors studied this property in different varieties. For example Ballico, Sommese and others studied k- very ampleness and k-spanndness in projective surfaces (See [BS94], [DR96] and [BS90]).

In the case of generic $(1,d)$ polarization of a g-dimensional abelian variety (X,L), Debarre, Hulek and Spandaw in [DHS94] proved that the line bundle L is very ample for $d > 2^g$.

For abelian surfaces, Bauer and Szenberg ([BS97]) succeeded in giving a numerical condition to check if the given line bundle is k-very ample. In particular, they proved that if L is a primitive line bundle with $c_1^2(L) = 2d$ on an abelian surface with Picard rank 1, then L is k- very ample if and only if $d \geq 2k+3$. We will use new methods to give another proof of this.

On the other hand, the weaker notion of k-spanned was studied in [BFS89]. Terakawa also studied the issue of k-very ampleness and k-spanned, (see [Ter98a] and [Ter98b]). He proved that the k-very ampleness is equivalent to k-spannedness for a polarized abelian surface.

Theorem 1.4.1. *([Ter98b, Corollary 4.2]) Let S be a polarized abelian surface and L an ample line bundle on it, then L is k-very ample if and only if L is k-spanned.*

Terakawa in [Ter98b] gave a necessary and sufficient condition for a line bundle to be k-very ample which is $c_1^2(L) \geq 4k+6$ but these depend on the existence of an effective divisor D satisfying the following

$$2\sqrt{(2k+3)(p_a(D)-1)} \leq L \cdot D \leq 2p_a(D)+k-1 \leq 2k+1,$$

where $p_a(D) = 1 - \chi(\mathcal{O}_D)$ is the arithmetic genus of D.

Here are some of classical results on the very ampleness of line bundles:

Theorem 1.4.2. *If L is an ample line bundle on an abelian variety V, then*

1. *L^2 is 0-very ample (see [PP04], §.1),*
2. *if L is a base-point free, then L^2 is very ample (Ohbuchi's Theorem[Ohb87]),*
3. *L^3 is very ample (Lefschetz's Theorem, [BL04, Cor. 4.5.3]).*

We will exploit the close relationship between k-very-ampleness and the IT condition of a certain sheaf with respect to a certain Fourier-Mukai transform Φ. These allow us to associate sheaves $\Phi^i(E)$ to a sheaf E. Then a sheaf E is IT_j if for each $i \neq j$, $\Phi^i(E) = 0$.

More precisely,

Proposition 1.4.3. *If (S, Φ) is an abelian variety and L is IT_0, then following are equivalent:*

1. *L is k-very ample.*

2. *$L \otimes \mathcal{I}_X$ is WIT_0 (and hence IT_0) for all $X \in \mathrm{Hilb}^{k+1} S$.*

Proof. See Proposition 3.1.3. □

Using the Fourier-Mukai transform, Popa and Pareschi introduced the notion of M-regularity (see [PP03] and [PP04]) which gives a condition on $\mathrm{Coh}(V)$ weaker than the Index Theorem (see below 1.5.13). If V is an abelian variety and $F \in \mathrm{Coh}(V)$, we say that F is M-regular (or Mukai regular) if $\mathrm{codim}(\mathrm{Supp}(\Phi^i F)) > i$ for each $i > 0$, where $\mathrm{Supp}(F)$ means the scheme theoretic support of the sheaf F. In their papers, they extended a relation between the notion of M-regularity and "weak index theorem" conditions.

This thesis will be concerned with the notation of critical value $\phi(L) = k$ for an ample line bundle L on an abelian surface S such that L is k-very ample but not $k+1$-very ample ([AM16]) using Bridgeland's stability conditions and Fourier-Mukai techniques. We are particularly interested in the case where the nth power of the polarization line bundle and then we turn of ϕ as a function of $n \in \mathbb{N}$.

A related notation is discussed in [BS97] where a special case of it is related to the Seshadri constants.

1.5 Key Tools

In this section we will go through the main two tools that we use in this thesis: Fourier-Mukai transforms and Bridgeland's stability conditions. Although we will be using them in the relatively specific setting of the derived categories of coherent sheaves on an abelian surface, these can be defined more abstractly for triangulated categories. We will need a technical lemma which is best viewed in this situation.

Triangulated Categories

Definition 1.5.1. *If X is a variety then the bounded derived category of coherent sheaves on X is denoted by $\mathcal{D}^b(X)$. For exact functor $F : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$, an object $E \in \mathcal{D}^b(X)$ is F-WIT_i if and only if $H^j(F(E)) = 0$ for all $j \neq i$.*

Definition 1.5.2. *For any $E \in \mathcal{D}(X)$ we define the Mukai vector $\boldsymbol{v}$ by:*

$$\boldsymbol{v} = \mathrm{ch}(E)\sqrt{td(X)} \in H^{\boldsymbol{ev}}(X, \mathbb{Q}),$$

where $td(X) = \dfrac{c_1(X)}{1-\exp(-c_1(X))}$ is the Todd class and $H^{\boldsymbol{ev}}(X,\mathbb{Q}) = H^0(X,\mathbb{Q}) \oplus H^2(X,\mathbb{Q}) \oplus H^4(X,\mathbb{Q})$.

For abelian surfaces $\mathrm{td}(X) = 1$ and $\mathrm{ch}(E) = (r(E), c_1(E), \chi(E))$ by the Riemann-Roch Theorem, and $\mathrm{ch}(E)$ is integral.

If $\boldsymbol{v} = \oplus \boldsymbol{v}_i \in H^{\boldsymbol{ev}}(X,\mathbb{Q})$, then $\boldsymbol{v}^\vee := \oplus(-1)^i \boldsymbol{v}_i$. Now we recall the definition of positive Mukai vector [Yos01, Def. 0.1] on a surface:

Definition 1.5.3. *A vector $\boldsymbol{v} = (r, c_1\ell, \chi)$ is said to be* ***a positive vector*** *($\boldsymbol{v} > 0$) if either*

- *$r > 0$, or*
- *$r = 0$ and c_1 is effective, or*
- *$r = 0$, $c_1 = 0$ and $\chi > 0$.*

Define

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \int \boldsymbol{v}^* \boldsymbol{w} = \chi(\boldsymbol{v}, \boldsymbol{w}).$$

Definition 1.5.4. *A Mukai vector $\boldsymbol{v} = (r, c_1\ell, \chi)$ is said to be* ***isotropic*** *if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$.*

Definition 1.5.5. *[GM03] Let T be an additive category with an additive equivalence $f : T \to T$. We say that it is pre-triangulated if there is a is a collection $\mathcal{D}$ of triangle diagrams called distinguished triangles satisfying the axioms (TR1)-(TR4) below:*

(TR1)

1. *Any triangle is in $\mathcal{D}$ if it is isomorphic to distinguished triangle.*
2. *For any object a of T, the triangle $a \xrightarrow{id_a} a \to 0 \to a[1]$ is in $\mathcal{D}$.*
3. *For each morphism $a \to b$ of objects of T, there is a triangle of $\mathcal{D}$ of the form*

$$a \to b \to c \to a[1],$$

think of c as b/a.

(TR2) If $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} a[1]$ is in $\mathcal{D}$ then so is $b \xrightarrow{g} c \xrightarrow{h} a[1] \xrightarrow{-f[1]} b[1]$

(TR3) Any commuting diagram

$$\begin{tikzcd}
a \arrow[r] \arrow[d, "f"] & b \arrow[r] \arrow[d] & c \arrow[r] & a[1] \arrow[d, "f[1]"] \\
a' \arrow[r] & b' \arrow[r] & c' \arrow[r] & a'[1]
\end{tikzcd}$$

where the rows are in $\mathcal{D}$ can be completed to a map of triangle diagrams.

(TR4) If $a \xrightarrow{f} b$ and $b \xrightarrow{g} c$ are two composible maps, and $a \xrightarrow{f} b \to a/b$, $b \xrightarrow{g} c \to b/c$ and $a \xrightarrow{gf} c \to a/c$ are in $\mathcal{D}$, then there is atriangle $a/b \to c/a \to c/b \to b/a[1]$ in $\mathcal{D}$ which is compatible with three diagrams in $\mathcal{D}$ determined by $a \to b$, $a \to c$ and $b \to c$ in (TR3).

Definition 1.5.6. *If E, F are two objects in $\mathcal{D}^b(X)$. Then define*

$$\mathrm{Ext}^i(E,F) = \mathrm{Hom}_{\mathcal{D}^b(X)}(E, F[i]).$$

Proposition 1.5.7. $\mathrm{Ext}^1(E,F)$ *is the group of equivalence classes of short exact sequences*

$$[G] : 0 \to F \to G \to E \to 0$$

under equivalent relation $[G] \sim [G']$ if there exists commutative diagram

$$\begin{tikzcd}
 & & G \arrow[dr] \arrow[dd] & & & \\
0 \arrow[r] & F \arrow[ur] \arrow[dr] & & E \arrow[r] & 0 \\
 & & G' \arrow[ur] & & &
\end{tikzcd}$$

in $\mathrm{Coh}(X)$.

Proof. See[GM03, p.184]. □

The following is a useful technical lemma that we shall need in different places in Chapter 4:

Lemma 1.5.8. *Suppose $\Phi : \mathcal{D}(X) \to \mathcal{D}(X)$ is fully faithful. Let $A \to B \to C \xrightarrow{\alpha} A[1]$ be a triangle in $\mathcal{D}(X)$. Further, suppose*

- *C is Φ-WIT_i,*
- *$\Phi^j(A) = 0$ for all j unless $j = i$ or $i+1$, and*
- *$\mathrm{Ext}^1(\Phi^i(C), \Phi^i(A))$=0,*

then $\Phi^i(\alpha) : \Phi^i(C) \to \Phi^{i+1}(A)$ is a non-zero map if and only if $\alpha \neq 0$.

Proof. Apply Φ to a sequence $A \to B \to C \to A[1]$, then we get

$$0 \to \Phi^i(A) \to \Phi^i(B) \to \Phi^i(C) \xrightarrow{\delta} \Phi^{i+1}(A) \to \Phi^{i+1}(B) \to 0$$

where $\delta = \Phi^i(\alpha) \in \mathrm{Hom}(\Phi^i(C), \Phi^{i+1}(A))$ and $\alpha \in \mathrm{Hom}(C, A[1])$ is the map of the triangle. Then we have a distinguished triangle

$$\tau_{<i+1}\Phi(A) \to \Phi(A) \to \tau_{\geq i+1}\Phi(A). \tag{1.5.1}$$

From our assumption $\Phi^j(A) = 0$ for all j unless $j = i$ or $i+1$, then $\tau_{\geq i+1}\Phi(A) = \Phi^{i+1}(A)[-i-1]$. Shift (1.5.1) by $i+1$ and apply $\mathrm{Hom}(\Phi^i(C), -)$ then we obtain the exact sequence

$$\mathrm{Hom}(\Phi^i(C), \tau_{<i+1}\Phi(A)[i+1]) \to \mathrm{Hom}(\Phi^i(C), \Phi(A)[i+1]) \to \mathrm{Hom}(\Phi^i(C), \Phi^{i+1}(A)).$$

Since $\mathrm{Hom}(\Phi^i(C), \tau_{<i+1}\Phi(A)[i+1]) = Ext^1(\Phi^i(C), \Phi^i(A))$, then $\mathrm{Hom}(\Phi^i(C), \tau_{<i+1}\Phi(A)[i+1]) = 0$. Therefore, $\mathrm{Hom}(\Phi^i(C), \Phi(A)[i+1])$ injects to $\mathrm{Hom}(\Phi^i(C), \Phi^{i+1}(A))$ and since Φ is fully faithful,

$$\begin{aligned}\mathrm{Hom}(C, A[1]) &= \mathrm{Hom}((C)[i], (A)[i+1]) \\ &\cong \mathrm{Hom}(\Phi(C)[i], \Phi(A)[i+1]) \\ &\hookrightarrow \mathrm{Hom}(\Phi^i(C), \Phi^{i+1}(A)),\end{aligned}$$

sending α to δ. Hence $\delta \neq 0$ if and only if $\alpha \neq 0$. □

Fourier-Mukai Transforms

In this section we will recall some basic definitions about the Fourier-Mukai functor, see [Muk81], [Huy06] and [YY] for more details.

Let V and W be smooth projective surfaces. Consider the flat projections $\pi : V \times W \to V$ and $\hat{\pi} : V \times W \to W$. Let $\mathbb{E}$ be any object of $\mathcal{D}(V \times W)$, where $\mathcal{D}(V \times W)$ denotes the derived category of bounded complexes of coherent sheaves on $V \times W$.

Definition 1.5.9. *The* ***Fourier-Mukai functor*** $\Phi_{\mathbb{E}}$ *is the functor*

$$\Phi_{\mathbb{E}} : \mathcal{D}(V) \to \mathcal{D}(W) \tag{1.5.2}$$

which takes $A^{\cdot}$ *into* $R\pi_*(\hat{\pi}^* A^{\cdot} \overset{L}{\otimes} \mathbb{E})$ *and we call* $\mathbb{E}$ *a Fourier-Mukai kernel. Denote its direct image by* $\Phi^i_{\mathbb{E}} = R^i\Phi_{\mathbb{E}}$ *for* $i \geq 0$. *When* $\mathbb{E}$ *is understood we just write* Φ *for* $\Phi_{\mathbb{E}}$. *If* $\Phi_{\mathbb{E}}$ *is an equivalence of categories then we call it a* ***Fourier-Mukai transform****.*

Example 1.5.10. *If the Fourier-Mukai transform induced by* $\mathcal{P}$ *the Poincaré bundle on an abelian surface* $V = S$. *Then* Φ *has a quasi-inverse given (up to shift) by the transform*

$$\hat{\Phi} : \mathcal{D}(\hat{S}) \to \mathcal{D}(S) \tag{1.5.3}$$

with kernel $\hat{\mathcal{P}} \in \mathcal{D}(\hat{S} \times S)$, *where* $\hat{\mathcal{P}} = s^*\mathcal{P}$ *and* $s : S \times \hat{S} \to \hat{S} \times S$ *is* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Until further notice, we shall only consider the Fourier-Mukai in Example 1.5.10. Now we need to recall some basic definitions

Definition 1.5.11. *Recall, an object* E *satisfies* Φ*-*WIT_n *(Week Index Theorem) or just* WIT_n *if* $\Phi^i(E) = 0$ *for all* $i \neq n$ *and we call* n *the index of* E *and it is denoted by* $\boldsymbol{i}(E) = n$.

Example 1.5.12. *(see [HP05]) Using Example 1.3.5, we have a Fourier-Mukai transform for any elliptic curve* $\mathcal{C} \cong \hat{\mathcal{C}}$

$$\Phi : \mathcal{D}(\mathcal{C}) \to \mathcal{D}(\mathcal{C}),$$

given by:

$$\mathcal{P} = \mathcal{O}(\Delta - X \times \{e\} - \{e\} \times X).$$

Up to shift all Fourier-Mukai transforms are given by the universal sheaf $\mathbb{E} \to \mathcal{C} \times \mathcal{C}$ *parametrizing vector bundles over* $\mathcal{C}$ *of rank* p *and degree* q *such that* $\gcd(p, q) = 1$.

In fact, $E|_{\mathcal{C}\times\{x\}} = E_x$ *satisfies for all* $x \in \mathcal{C}$, $E_x = E_e \otimes \mathcal{P}_x$. *Hence,* F *is* $\Phi_{\mathbb{E}}$*-WIT*$_i$ *if and only if* $F \otimes E_e$ *is* $\Phi_{\mathcal{P}}$*-WIT*$_i$.

Definition 1.5.13. *If* $\mathcal{P}$ *parametrizes bundles, then an object* E *satisfies* IT_n *(Index Theorem) if* $H^i(E \otimes \mathcal{P}_y) = 0$ *for all* $y \in W$ *and* $i \neq n$. *In which case,* $\Phi^n(E)$ *is a locally free sheaf.*

Remark 1.5.14. *([Mum74])*

- *If an ample line bundle L on an abelian variety has no higher cohomology, then it is* IT_0
- *Any sheaf which is* WIT_0 *is automatically* IT_0 *by the semi-continuity theorem [Mum70].*

Proposition 1.5.15. *([Muk81]) If* E *is a sheaf with the Chern character* $\mathrm{ch}(E) = (r, c\ell, \chi)$, *then the Chern character of* $\Phi(E)$ *is* $(\chi, -\hat{c}, r)$, *where* $\hat{c}$ *is the dual of* $c_1(E)$.

Theorem 1.5.16. *(See [Theorem 2.2.[Muk81]) There are isomorphisms of functors:*

$$\Phi \circ \hat{\Phi} \cong (-1_{\hat{S}})^*[-g]$$

and

$$\hat{\Phi} \circ \Phi \cong (-1_S)^*[-g]$$

where $[-g]$ *denotes shift the complex g places to the right.*

From this we can define:

Definition 1.5.17. *For* $E \in \mathrm{Coh}(X)$, *there exists a spectral sequence with* E_2 *term*

$$E_2^{p,q} = \hat{\Phi}^p(\Phi^q(E))$$

where $p, q \in \mathbb{Z}$ *converges to*

$$E_{\infty}^{p+q} = \begin{cases} (-1)^* E & \text{if } p+q=2 \\ 0 & \text{otherwise} \end{cases}$$

For $E \in \mathrm{Coh}(X)$, we can describe the spectral sequence $E_2^{p,q}$ as follows (see [Mac96b]). Since $\Phi^q(E) = 0$ for $q < 0$ and $q > 2$, then we have a three by three diagram

$$\begin{tikzcd}
\hat{\Phi}^0(\Phi^2(E)) \arrow[drr, "d_2^{0,2}"] & 0 & 0 \\
\hat{\Phi}i^0(\Phi^1(E)) \arrow[drr, "d_2^{0,1}"] & \hat{\Phi}^1(\Phi^1(E)) & \hat{\Phi}^2(\Phi^1(E)) \\
0 & 0 & \hat{\Phi}^2(\Phi^0(E))
\end{tikzcd}$$

such that there are

- $d_2^{0,2} : \hat{\Phi}^0(\Phi^2(E)) \to \hat{\Phi}^2(\Phi^1(E))$ is surjective, and
- $d_2^{0,1} : \hat{\Phi}i^0(\Phi^1(E)) \to \hat{\Phi}^2(\Phi^0(E))$ is injective.

Then we obtain two long exact sequences:

$$0 \to D \to (-1)^* E \to \hat{\Phi}^0(\Phi^2(E)) \to \hat{\Phi}^2(\Phi^0(E)) \to 0$$

$$0 \to \hat{\Phi}i^0(\Phi^1(E)) \to \hat{\Phi}^2(\Phi^0(E)) \to D \to \hat{\Phi}^1(\Phi^1(E)) \to 0$$

where D is some auxiliary sheaf. If $\hat{\Phi}^0(\Phi^2(E)) = 0$, we obtain a single exact sequence

$$0 \to \hat{\Phi}^0(\Phi^1(E)) \to \hat{\Phi}^2(\Phi^0(E)) \to (-1)^* E \to \hat{\Phi}^1(\Phi^1(E)) \to 0.$$

If $\hat{\Phi}^2(\Phi^0(E)) = 0$, we obtain a single exact sequence

$$0 \to \hat{\Phi}^1(\Phi^1(E)) \to (-1)^* E \to \hat{\Phi}^0(\Phi^2(E)) \to \hat{\Phi}^2(\Phi^1(E)) \to 0.$$

Example 1.5.18. *(See [Mac11])*

Suppose $X = \mathcal{C}$ is an elliptic curve. For $E \in \mathrm{Coh}(\mathcal{C})$, spectral sequence gives a single short exact sequence:

$$0 \to \hat{\Phi}^1_{\mathcal{P}}(\Phi^0_{\mathcal{P}}(E)) \to E \to \hat{\Phi}^0_{\mathcal{P}}(\Phi^1_{\mathcal{P}}(E)) \to 0.$$

Bridgeland's Stability Conditions

We begin this section with a short summary of Bridgeland's stability conditions (see [Bri08] and [Bri07] for further details). Throughout this section S is an abelian surface such that $\mathrm{NS} = \langle \ell \rangle$ and L is a polarization line bundle on S and has $c_1(L) = \ell$ where $\ell^2 = 2d$.

If $\mathcal{A}$ is an abelian category, the definition of a stability function can be stated as follows.

Definition 1.5.19. *A stability function on $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that*

1. *For each $0 \neq E \in \mathcal{A}$,*

   $$Z(E) = m(E)e^{i\pi\phi(E)} \text{for some } m(E) \in \mathbb{R}_{>0}.$$

   The real number $\phi(E) \in (0,1]$ is called the phase of the object E and it is given by $\phi(E) = (1/\pi)\arg Z(E)$. We call $\sigma = (\mathcal{A}, Z)$ Bridgeland stability condition and Z is the central charge.

2. *There is a unique Harder- Narasimhan filtration for E*

   $$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E,$$

   in $\mathcal{A}$ such that the factor $F_i = E_i/E_{i-1}$ is a semistable object for $i = 1, \ldots, k$ satisfying

   $$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_k).$$

3. *(Support property, [KS08]): For each* $0 \neq E \in \mathcal{A}$,

$$\|E\| \leq C|Z(E)|$$

where C *is a constant and* $\|.\|$ *is a fixed norm on* $\mathbb{C} \otimes \mathbb{R}$.

We can construct examples of these when $T = \mathcal{D}^b(S)$ and S is abelian surface as follows:

Definition 1.5.20. *Let* $\mathcal{A}$ *be an abelian category and* $(\mathcal{T}, \mathcal{F})$ *a pair of full subcategories of* $\mathcal{A}$ *such that* $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ *for* $T \in \mathcal{T}$ *and* $F \in \mathcal{F}$. *We call* $(\mathcal{T}, \mathcal{F})$ *a torsion pair if every object* $E \in \mathcal{A}$ *fits into a short exact sequence*

$$0 \to T \to E \to F \to 0$$

for some $T \in \mathcal{T}$ *and* $F \in \mathcal{F}$ *and we call the objects of* $T \in \mathcal{T}$ *and* $F \in \mathcal{F}$ *torsion and torsion-free, respectively.*

Example 1.5.21. *Define for any* $\beta \in \mathbb{R}$ *and* $w \in \mathrm{NS}(S)$ *the sub category* $\mathcal{A}_{\beta,w}$ *of* $\mathcal{D}(S)$ *by the following:*

$$\mathcal{F}_{\beta,w} = \{E \in \mathrm{Coh}_S \,|\, E \textit{ is torsion-free and } \mu_+(E) \leq \beta.w\},$$

$$\mathcal{T}_{\beta,w} = \{E \in \mathrm{Coh}_S \,|\, E \textit{ is torsion or } \mu_-(E/\mathrm{tors}(E)) > \beta.w\},$$

where $\mu_+(E)$ *is the slope of the largest slope* μ*-destabilizing subsheaf of* E *and* $\mu_-(E)$ *is the slope of the lowest slope* μ*-destabilizing quotient of* E.

Example 1.5.22. *If* $\mathcal{A} = \mathrm{Coh}(S)$, *define for any* $s \in \mathbb{R}$ *the following:*

$$\mathcal{F}_s = \{E \in \mathrm{Coh}_S \,|\, E \textit{ is torsion-free and } \mu_+(E) \leq 2ds\},$$

$$\mathcal{T}_s = \{E \in \mathrm{Coh}_S \,|\, E \textit{ is torsion or } \mu_-(E/\mathrm{tors}(E)) > 2ds\},$$

We set

$$\mathcal{A}_s = \{A \in \mathcal{D}(\mathcal{A}) | A^i = 0, i \notin \{0, -1\}, H^{-1}(A) \in F_s, H^0(A) \in T_s\}.$$

Definition 1.5.23. *Take a pair $s, t \in \mathrm{NS}(S) \otimes \mathbb{R}$ with $t \in \mathrm{Amp}(S)$. The* ***stability function*** *on $\mathcal{A}_s$ is a group homomorphism $Z_{s,t} : K(S) \to \mathbb{C}$ which takes the Chern character* $\mathrm{ch}(A)$ *for each $A \in \mathcal{A}_s$ into*

$$Z_{s,t}(A) = -\int e^{-(s+ti)} \cdot \mathrm{ch}(A)$$

Theorem 1.5.24. *([AB11], [Bri08]) If S is an abelian surface, the pair $(Z_{s,t}, \mathcal{A}_s)$ is stability condition.*

If the Picard rank 1 and $\mathrm{NS} = \langle \ell \rangle$, the Chern character $\mathrm{ch}(E)$ of E is $(r, c\ell, m)$ where $r, c \in \mathbb{Z}$. In the case of abelian surfaces we have $m = \chi(E)$ is an integer number. Also we write s as $s\ell$ and t as $t\ell$ where $t > 0$.

In this case,

$$Z_{s,t}((r, c\ell, \chi)) = -\chi + 2dcs + dr(t^2 - s^2) + 2tdi(c - rs) \tag{1.5.4}$$

We let $\mathbb{H} = \{(s,t) \in \mathbb{R}\ t > 0\}$ be the upper half plane. The slope $\mu_{s,t}(A)$ of A is given by:

$$\mu_{s,t}(A) = -\frac{\mathrm{Re}(Z_{s,t}(A))}{\mathrm{Im}(Z_{s,t}(A))} \tag{1.5.5}$$
$$= \frac{\chi - 2dcs - dr(t^2 - s^2)}{2td(c - rs)}. \tag{1.5.6}$$

Definition 1.5.25. *An object $E \in \mathcal{A}_s$ is said to be σ_t-**stable** (respectively, σ_t-**semistable**) if for all proper injections $F \to E$ in $\mathcal{A}_s$ we have*

$$\mu_{s,t}(F) < \mu_{s,t}(E) \quad (\mu_{s,t}(F) \leq \mu_{s,t}(E), \textit{respectively}).$$

Definition 1.5.26. *Let A be a non-trivial object in an abelian category $\mathcal{A}$. We said that A is a* ***minimal object*** *if and only if any surjection $A \twoheadrightarrow B$ with $B \neq 0$ is an* *isomorphism. Equivalently, A is minimal if every injection $C \hookrightarrow A$ with $C \neq 0$ is an isomorphism. In other words, A has no proper subobjects.*

Remark 1.5.27. *We say that A is not stable if there is an object $F \hookrightarrow A$ in $\mathcal{A}_s$ such that $\mu_{s,t}(F) > \mu_{s,t}(A)$. The critical values of s and t when $\mu_{s,t}(F) = \mu_{s,t}(A)$ are semicircles with centers on s-axis and we call them* ***walls****.*

Consider the case where A is a sheaf in $\mathcal{A} \cap \mathrm{Coh}(S)$. Suppose $B \in \mathcal{A}_s$ with $\mathrm{ch}(B) = (r_1, c_1\ell, \chi_1)$ destabilizes A, then we have a short exact sequence in $\mathcal{A}_s$:

$$0 \to B \to A \to Q \to 0 \tag{1.5.7}$$

If we take the cohomology of (1.5.7) we see that $H^{-1}(B) = 0$. Then $B \in \mathcal{T}_s$ and $c_1 > r_1 s$. Notice also that $H^{-1}(Q) \in \mathcal{F}_s$ is torsion-free.

From the definition of stable objects and since B destabilizes A, then $\mu_{s,t}(B) - \mu_{s,t}(A) > 0$. Therefore

$$\frac{\chi_1 - 2dc_1 s + dr_1(s^2 - t^2)}{2td(c_1 - r_1 s)} - \frac{\chi - 2dcs + dr(s^2 - t^2)}{2td(c - rs)} \geq 0. \tag{1.5.8}$$

The numerator of (1.5.8) is the following

$$\begin{aligned}&(\chi_1 - 2dc_1 s + dr_1(s^2 - t^2))(c - rs) - (\chi - 2dcs + dr(s^2 - t^2))(c_1 - r_1 s)\\ &= (c - rs)\chi_1 - (c_1 - r_1 s)\chi + d(c_1 r - cr_1)(s^2 - t^2).\end{aligned}$$

Hence the condition is

$$(c - rs)\chi_1 - (c_1 - r_1 s)\chi + d(c_1 r - cr_1)(s^2 - t^2) \geq 0 \tag{1.5.9}$$

and the equality gives us the equations of the walls. We shall often be interested in the case when $s = 0$. In this case, the destabilizing condition becomes

$$c\chi_1 - c_1\chi + dt^2(cr_1 - c_1 r) \geq 0. \tag{1.5.10}$$

Recall the definition of a slicing

Definition 1.5.28. *([Bri07]) If $\mathcal{D}$ is a triangulated category, a slicing $\mathcal{P}$ of $\mathcal{D}$ consists of full additive subcategory $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ satisfying the following:*

- $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$, *for all* $\phi \in \mathbb{R}$.
- *For any* $\phi_1 > \phi_2$ *and* $A_i \in \mathcal{P}(\phi_i)$, *then* $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$.
- *For each* $0 \neq E \in \mathcal{D}$ *there are a sequence of real numbers* $\phi_1, \phi_2, \dots, \phi_n$ *such that* $\phi_{j+1} < \phi_j$ *and* $j = 1, \dots, n-1$ *and a set of triangles*

$$\begin{tikzcd}
E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \dots \arrow[r] & E_{n-1} \arrow[rr] & & E_n \arrow[dl] \\
& A_1 \arrow[ul, dotted] & & A_2 \arrow[ul, dotted] & & & & A_n \arrow[ul, dotted] &
\end{tikzcd}$$

where $E_0 = 0$, $E_n = E$ *and* $A_i \in \mathcal{P}(\phi_i)$ *for* $i = 1, \dots, n$.

Knowledge of Bridgeland's stability conditions should allow us to deduce geometrical information about varieties as we will see later in Chapter 5.

1.6 k-Very Ample for Irreducible Abelian Surfaces

Let S be an abelian surface and L an ample line bundle on S with $c_1(L) = \ell$ and $\ell^2 = 2d$. The map ϕ takes each L^n from the ample cone on S into $k \in \mathbb{Z}_{\geq -1}$ (see Definition 3.1.4) such that L^n is k-very ample but not $(k+1)$-very ample.

For the principal polarization line bundle we prove the following.

Theorem 1.6.1. *Let (S, L) be a principally polarized abelian surface, then for $n \geq 3$ we have $\phi(L^n) = 2n - 4$.*

Proof. See Proposition 3.2.2, Corollary 3.2.3 and Lemma 3.2.9. □

We also generalize this theorem for a polarization line bundle of degree $d \geq 1$ and we get the value of $\phi(L^n)$ for $n \geq 1$.

In particular we give a proof of the special case when $n = 1$:

Proposition 1.6.2. *Let (S, L) be a polarized abelian surface of degree d, then*

$$\phi_L(1) = \left\lfloor \frac{d-3}{2} \right\rfloor$$

Proof. See Proposition 3.2.12. □

For the large value of n we will use a different method to find ϕ. In this technique we will use the relation between IT_0 property and the k-very ampleness that we mentioned above, and also compute walls in the stability space associated to the Chern character $\mathrm{ch} = (1, n\ell, (n-1)^2 d + d + 1)$. That allows us to bound ϕ from above and then prove that the bound is sharp.

To show that the upper bound is sharp, we will need only to show that the moduli space of Bridgeland stable objects of this Chern character $\mathcal{M}^{\sigma_{s,t}}_{(1,n\ell,n^2d-k)}$ is IT_0 for all $t \gg 0$ and $k = 2d(n-1)-1$. i.e. for all $t \gg 0$, each E representing an object of $\mathcal{M}^{\sigma_{s,t}}_{(1,n\ell,n^2d-k)}$ satisfies IT_0. We do that by observing that any sheaf which does not remain Bridgeland stable for all $t > 0$ must be IT_0 by induction. Otherwise, the dual of the sheaf remains Bridgeland stable for all $t > 0$ and so the Fourier-Mukai transform must be a sheaf hence locally-free as it is of the form $\Phi^0(E)$. If F with $\mathrm{ch}(F) = (r(F), c_1(F), \chi(F))$ destabilizes E, we prove the following

Theorem 1.6.3. *Let (S, ℓ) be a polarized abelian surface such that $\mathrm{NS}(S) = \langle \ell \rangle$ and $\ell^2 = 2d$, and $E \in \mathcal{M}^{\sigma_{s,t}}_{(1,n\ell,n^2d-k)}$. If F is a maximal destabilizer of E, we have*

1. *F is a torsion-free sheaf.*

2. *F is Bridgeland-stable and so Bogomolov holds (as S is abelian).*

3. $r(F) = 1$.

4. $c_1(F) = (n-1)\ell$.

5. $|\mathrm{Sing}(F)| \leq 2d(n-2) - 1$.

This gives rise to the following

Theorem 1.6.4. *Let (S, L) be a polarized abelian surface of degree d, then for $n \geq 1$ we have $\phi(L^n) = 2d(n-1) - 2$.*

In Chapter 4, we study the k-very ampleness for principal polarization abelian surfaces with Picard rank greater than 1. In particular, the the product case when (S, ℓ) is a principal polarized abelian surface such that S is a product of two elliptic curves $\mathcal{C}_1$ and $\mathcal{C}_2$ corresponding to line bundles L_1 and L_2 on S.

Let $L = L_1 \otimes L_2$ and abbreviate $n_1\ell_1 + n_2\ell_2$ to (n_1, n_2) where $\ell_i = c_1(L_i)$. We use induction to prove the following

Theorem 1.6.5. *Let $L = L_1^{n_1} \otimes L_2^{n_2}$, then the critical k-very ampleness $\tilde{\phi}(n_1, n_2) = \min(n_1, n_2) - 2$.*

We generalize this theorem when $\mathcal{C}_1 \cdot \mathcal{C}_2 = d$ and we get

Theorem 1.6.6. *Let $\tilde{L} = \tilde{L}_1^{n_1} \otimes \tilde{L}_2^{n_2}$, then $\tilde{\phi}(n_1, n_2) = d\min(n_1, n_2) - 2$.*

There are some special cases that we discover in the same chapter which are shown below

Example 1.6.7. *If $L = L_1 \otimes L_2^2$ (or $L = L_1^2 \otimes L_2$), then $\phi(1,2) = -1$ (and $\phi(2,1) = -1$) and L is not k-very ample for any integers $k \geq 0$.*

The analysis of k-very ampleness naturally leads us to understand the moduli space of torsion sheaves of Chern character $\mathrm{ch}(T) = (0, \ell, \beta)$. In chapter six we look in more detail at the walls for this Chern character. Maciocia in [Mac14] studies the rank zero case in the following proposition

Proposition 1.6.8. *[Mac14, Proposition 4.1] If (S,ℓ) is a polarized abelian surface with $\ell^2 = 2d$ and $\rho(S) = 1$ then there are no walls for $\boldsymbol{v} = (0,\ell, n\ell^2) = (0,\ell,2dn)$ for all $n \in \mathbb{Z}$.*

He also in the same paper proved that the line $s = C_0 = \dfrac{\chi(T)}{\ell^2.c_1(T)}$ must intersects all the walls of T.

Theorem 1.6.9. *[Mac14, Theorem 3.11] Let $s,t \in \mathrm{NS}(S)\otimes\mathbb{Q}$ with $s \in \mathrm{Amp}(S)$. and $u \in \mathbb{R}$. Fix a Chern character* ch *of a μ-semistable sheaf or a torsion sheaf. In the half-plane $\Pi_{s,t,u}$ there are real numbers C_0 and R_0 such that all of the walls corresponding to* ch *are contained in the semi-circle with centre $(C_0,0)$ and radius R_0. In particular, the radii of the walls are bounded above by R_0.*

Therefore we will fix $s = C_0 = \beta/2d$. Let A be a Bridgeland stable destabilizer of T in $\mathcal{A}_{\frac{\beta}{2d}}$, then there is a short exact sequence

$$0 \to A \to T \to Q \to 0 \tag{1.6.1}$$

in $\mathcal{A}_{\beta/2d}$ such that $\mu_{s,t}(A) \geq \mu_{s,t}(T)$. We will allocate all destabilizers when $c_1(A) = \ell$ of this Chern character. In particular, we fix $s = \beta/2d$ which is the line that intersects all the walls of T. Early in this chapter we use duality and tensoring (1.6.1) with L^N for $N \geq 1$ to prove the following, let $W(\beta)$ denote the number of walls corresponding to the Chern character $(0,\ell,\beta)$

Proposition 1.6.10. *If T is a torsion with* $\mathrm{ch}(T) = (0,\ell,\beta)$*, then*

1. *$W(\beta + 2Nd) = W(\beta)$, for $N \geq 1$,*
2. *$W(-\beta) = W(\beta)$.*

Proof. See Lemma 6.2.3 and 6.2.4. □

However, if A has the Chern character $\mathrm{ch}(A) = (r(A), c_1(A), \chi(A))$, then we prove the following

Proposition 1.6.11. *If T and A are as above, and $c_1(A) = \tilde{c}_1(A)\ell$, then*

- $\tilde{c}_1(A) = \left\lfloor \dfrac{r(A)\beta}{2d} \right\rfloor + 1,$
- $\tilde{c}_1(A)\beta - \frac{\beta^2 r(A)}{4d} < \chi(A) \leq \min(\tilde{c}_1^2(A)d/r(A), \tilde{c}_1(A)\beta)$

Proof. See Propositions 6.2.6 and 6.2.11. □

Furthermore, Lo and Qin in [LQ14] Prove the follows

Theorem 1.6.12. *Let $(s,t) \in \mathrm{NS}(S) \otimes \mathbb{R}$. For any Chern character $\boldsymbol{v}$*

1. *The set of mini-walls of $\boldsymbol{v}$ is locally finite.*
2. *There is t' such that there are no walls of $\boldsymbol{v}$ for all $t \in [t', \infty)$.*

In other words, there are no walls for t sufficiently large. Then the walls are bounded above by the biggest wall. In our case we show that for the biggest wall which has the largest radius, the first Chern character of destabilizers is always $c_1(A) = \ell$ (see Lemma 6.2.10).

Back to the paper [Mac14], it is shown that the number of walls for any Chern character of the form $(0, \ell, \beta)$ is finite. For the small value of β for example $\beta = 0, 1, 2$ or d we are able to count the number of walls by bounded the rank of destabilizers.

Theorem 1.6.13. *If T is a torsion with $\mathrm{ch}(T) = (0, \ell, \beta)$, then*

1. $W(0) = 0,$
2. $W(1) = d,$
3. $W(2) = \left\lfloor \dfrac{d}{2} \right\rfloor,$
4. $W(d) = \left\lceil \dfrac{d}{4} \right\rceil.$

Proof. See Lemma 5.2.2 and Propositions 6.4.1, 6.4.2 and 6.4.4. □

We denote the number of walls T when $\mathrm{ch}(A) = (r, c\ell, \chi)$ by $W(\beta, r, c)$ and we give a formula of this number which is given by

$$W(\beta, r, c) = \left\lceil \frac{c^2 d}{r} \right\rceil + \left\lceil \frac{\beta^2 r}{4d} \right\rceil - c\beta \tag{1.6.2}$$

On the other hand, as it is generally not possible to bound the rank of the destabilizer, we use a general Fourier-Mukai transforms to bound the radius of the walls. In §6.3, we follow the calculations of Maciocia and Piyaratne in [MP13] and of Yoshioka and Yanagida in [YY14] to give these transforms in two cases when β is odd and even. We find that the Fourier-Mukai transform $\Phi_{\beta/2d}$ preserves the rank and the first Chern character of $\mathrm{ch} = (o, \ell, \beta)$.

Chapter 2

Fourier-Mukai Transforms and Stability Conditions on Abelian Surfaces

2.1 Fourier-Mukai Transforms

Let V and W be smooth projective varieties. The Fourier-Mukai functor Φ is a type of exact equivalence between two derived categories $\Phi : \mathcal{D}(V) \to \mathcal{D}(W)$. Transforms of this type are very interesting in that they behave well under composition. For some $\mathbb{E} \in \mathcal{D}^b(V \times W)$, the Fourier-Mukai functor $\Phi_{\mathbb{E}}$ is given by

$$\Phi_{\mathbb{E}}(F) = R\pi_{*W}(L\pi_V^* F \overset{L}{\otimes} \mathbb{E})$$

with kernel $\mathbb{E}$.

There are a large class of these transforms for any abelian variety and they have been classified by Orlov. The following theorem by Orlov gives a relation between any functor and the one of Fourier-Mukai type.

Theorem 2.1.1. *(Orlov's Representation Theorem) Any fully faithful exact functor $\mathcal{D}^b(V) \to \mathcal{D}^b(W)$ is uniquely isomorphic to one of the form $\Phi_{\mathbb{E}}$ for some $\mathbb{E} \in \mathcal{D}^b(V \times W)$.*

Proof. See [Orl97, Theorem 2.18]. □

2.2 General Fourier-Mukai Transforms

Let X, Y be abelian varieties, $\hat{X}, \hat{Y}$ are their dual varieties respectively, and f an isomorphism between $X \times \hat{X}$ and $Y \times \hat{Y}$. We can write f as a matrix

$$f = \begin{pmatrix} x & y \\ w & z \end{pmatrix}$$

where $x : X \to Y$, $y : \hat{X} \to Y$, $w : X \to \hat{Y}$ and $z : \hat{X} \to \hat{Y}$. If the inverse of f has the form

$$f^{-1} = \begin{pmatrix} \hat{x} & -\hat{y} \\ -\hat{w} & \hat{z} \end{pmatrix}$$

we say that f is **isometric** (see [Orl02] and [Huy06]).

Note that $X \simeq \hat{\hat{X}}$ and $Y \simeq \hat{\hat{Y}}$ and $\hat{y}$ is defined as

$$\hat{y} : \hat{Y} \to \hat{\hat{X}} \xrightarrow{\simeq} X.$$

In the case of $X = Y$, let (X, L) be an abelian surface and $c_1(L) = \ell$ with $\ell^2 = 2d$ and $\mathrm{NS}(X) = \langle \ell \rangle$. We can give a much more explicit version of Orlov's theorem of a general Fourier-Mukai transformation Φ and the matrix description of its associated cohomological Fourier-Mukai transform introduced in [Mac96a] and [YY] going back to [Muk78].

Let E be a simple homogenous bundle with $\mathrm{ch}(E) = (x^2 a, -xy\ell, y^2 b)$ where a and b are coprime, $d = ab$. We get all Fourier-Mukai transforms if

$$\begin{pmatrix} ax & by \\ w & z \end{pmatrix} \in SL_2(\mathbb{Z}),$$

i.e. $axz - byw = 1$.

Following the calculations in [MP13] we can induce the cohomological Fourier-Mukai Transforms which are given by three by three matrix (a_{ij}) where

$$\begin{aligned} a_{ij} &= (-1)^{j-i} \sum_{k\in\mathbb{Z}} \binom{3-i}{k-1}\binom{i-1}{j-k} (\sqrt{a}x)^{-i-k+4}(\sqrt{b}y)^{k-1}(\sqrt{a}w)^{i-j+k-1}(\sqrt{b}z)^{j-k} \\ &= (-1)^{j-i} \sum_{k\in\mathbb{Z}} \binom{3-i}{k-1}\binom{i-1}{j-k} \sqrt{a^{3-j}b^{j-1}}\, x^{-i-k+4} y^{k-1} w^{i-j+k-1} z^{j-k} \end{aligned}$$

Then we get a function $\Phi^H_{\mathbb{E}} : H^{\boldsymbol{eV}}(V,\mathbb{Z}) \to H^{\boldsymbol{eV}}(\hat{V},\mathbb{Z})$ which is given by

$$\Phi^H_{\mathbb{E}} = \begin{pmatrix} ax^2 & -2xy\ell & by^2 \\ -xz\ell & axz+byw & -yw\ell \\ bw^2 & -2zw\ell & az^2 \end{pmatrix} \tag{2.2.1}$$

2.3 Stable Sheaves on Surfaces

Let L be an ample line bundle with $c_1(L) = \ell$ on a smooth projective variety V of dimension $\boldsymbol{g}$. In this section we will give a short review of the classical stability in the category of coherent sheaves $\mathrm{Coh}(X)$ on a scheme X (See [HL10] for more details).

Definition 2.3.1. *We define the **degree** of the coherent sheaf E by*

$$\deg(E) = c_1(E).\ell^{\boldsymbol{g}-1},$$

and the slope $\mu(E)$ by

$$\mu(E) = \frac{\deg(E)}{r(E)}.$$

Note that the degree and the slope are depend on the ample divisor L.

Definition 2.3.2. *A torsion-free sheaf E is μ-**stable** (μ-**semistable**) with respect to ℓ if for each subsheaf F of E we have*

$$\mu(F) < \mu(E) \quad (\mu(F) \leq \mu(E)).$$

If E is torsion-free sheaf, then there is a unique Harder- Narasimhan filtration for E

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E,$$

such that the factor $F_i = E_i/E_{i-1}$ is a μ-semistable sheaf for $i = 1, \ldots, k$ which satisfying

$$\mu^+ = \mu_1 > \cdots > \mu_k = \mu^-$$

where $\mu_i = \mu(F_i)$ and E is μ-semistable if and only if $\mu^+ = \mu^-$.

The following Lemma review some properties of stability and semi stability (see [OSS], p.164).

Lemma 2.3.3. *If V is a smooth projective variety, then*

- *Line bundles areμ- stable.*
- *E is μ-stable (μ-stable) if and only if E^* is.*

We need also to recall the Bogomolov Inequality for semi-stable sheaves. We will use it in various places.

Theorem 2.3.4. *(**Bogomolov's Inequality**) Let V be a smooth projective variety of dimension n and ℓ be an ample divisor on V. If E is a μ-semistable (with respect to ℓ) torsion sheaf of rank r on V, then*

$$(r-1)c_1^2(E).\ell^{n-2} \leq 2rc_2^2(E).\ell^{n-2}.$$

For the case of an abelian surface this reads:

$$2r(E)\chi(E) \leq c_1^2(E).$$

Proof. See [HL10, Theorem 7.3.1]. □

We will also need to consider a finer stability for sheaves:

Definition 2.3.5. *A torsion-free sheaf E is **Gieseker stable** (respectively **Gieseker semistable**) with respect to ℓ if for each subsheaf F we have*

$$P(F) < P(E) \quad (P(F) \leq P(E))$$

where $P(E) = \dfrac{\chi(E \otimes L^n)}{r(E)}$, is the reduced Hilbert polynomial.

Note 2.3.6. *If f, g are two polynomials, then $f \leq g$ if and only if $f(m) \leq g(m)$ for $m \gg 0$ and $f < g$ if and only if $f(m) < g(m)$ for $m \gg 0$.*

Definition 2.3.7. *If E is a pure sheaf, then there is a unique Harder- Narasimhan filtration for E*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E,$$

such that the factor $F_i = E_i/E_{i-1}$ is a semistable sheaf for $i = 1, \ldots, k$ which satisfying

$$P^+ = P_1 > \cdots > P_k = P^-$$

where $P_i = P(F_i)$.

If E is a torsion-free sheaf, then we have the following chain of implications (Lemma 1.2.13, [HL10])

$$E \text{ is } \mu\text{-stable} \Longrightarrow E \text{ is Gieseker-stable} \Longrightarrow E \text{ is Gieseker-semistable}$$

$$\Longrightarrow E \text{ is } \mu\text{-semistable.}$$

2.4 Fourier-Mukai Transforms and Stability

We can ask,

$$\text{if given sheaf } E, \text{ is } \Phi^i(E)\ \mu\text{-stable (or Gieseker-stable)?}$$

Generally the answer is no but these are some results in the positive direction,

Theorem 2.4.1. *[Mac96a, Corollary 3.4] If $\mu(E)=0$ and E is μ-stable then so is $\Phi^1(E)$.*

Bridgeland stability allows us to correct this in that if E is Gieseker-stable then $\Phi(E)$ is Bridgeland stable. In the case of the rank of a sheaf is one then we have a useful theorem

Theorem 2.4.2. *[Mac12, Theorem 11.1] If S is an abelian surface and X is a 0-dimensional with $|X|\geq 2$, then $\Phi^1(L\mathcal{I}_X)$ is μ-stable.*

2.5 Moduli Space of Stable Sheaves

Let S be an abelian surface over $\mathbb{C}$ and fix a polynomial $h\in\mathbb{Q}(t)$.

Definition 2.5.1. *The* ***moduli functor*** *is the functor*

$$\mathbb{M}_{\boldsymbol{S}}^h:\mathfrak{Sch}_{\mathbb{C}}\to\mathfrak{Set}$$

from $\mathfrak{Sch}_{\mathbb{C}}$ the set of all scheme of finite type X over $\mathbb{C}$ to the set of equivalent class $\mathfrak{Set}$ of X-flat sheaf $\mathscr{E}$ on $S\times X$ with $\boldsymbol{v}(E)=\boldsymbol{v}$ such that $\mathscr{E}_s$ is semi-stable with respect to h for all closed point $s\in S$.We say that two sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ are equivalent if there exists a line bundle $\mathscr{L}$ such that $\mathscr{E}_1\simeq\mathscr{E}_2\otimes\pi^*\mathscr{L}$ and π is a projection map $\pi:S\times X\to X$.

Theorem 2.5.2. *([Gie77], [Mar78], [Sim94]) The functor $\mathbb{M}_{\boldsymbol{S}}^h$ is corepresented by a coarse moduli space $\mathcal{M}^{GS}$ which is a projective variety.*

Definition 2.5.3. *We let $\mathcal{M}_{\mathrm{ch}}^{\mathrm{GS}}$ denote the moduli space of Gieseker semistable sheaves on S with Chern character ch, or more generally, Simpson semistable sheaves when the rank is zero.*

Definition 2.5.4. *The* ***virtual dimension of the moduli space*** *$\mathcal{M}_{(r,c\ell,\chi)}^{\mathrm{GS}}$ of Gieseker semistable sheaves with Chern character $\mathrm{ch}=(r,c\ell,\chi)$ is given by*

$$\dim\mathcal{M}_{(r,c\ell,\chi)}^{\mathrm{GS}}=2c^2d-2r\chi+2.$$

Yoshioka in [Yos01], proved that these spaces are non-empty exactly when this dimension is at least 2 (in other words, exactly when the Bogomolov inequality holds)

Theorem 2.5.5. *(Yoshioka) Let $\boldsymbol{v}$ be a Mukai vector such that $\boldsymbol{v} = m\boldsymbol{v}_0$, $m \in \mathbb{Z}$ and $\boldsymbol{v}_0$ is primitive positive vector, then the moduli space $\mathcal{M}_{\boldsymbol{v}} \neq \phi$.*

The case when the dimension is exactly 2 was proved by Mukai in [Muk78, Prop 6.22] and the remaining cases are dealt with in [Yos01, Thm 0.1]. We will need non-emptiness specifically for the cases where $c = \pm\ell$ and $\chi = 1$ or $\chi = 2$ which are studied in detail in [Yos01, §6].

Definition 2.5.6. *We shall say that the moduli space $\mathcal{M}^{GS}_{(r,c\ell,\chi)}$ satisfies IT_0 (respectively WIT_0) if and only if for each E representing an object of $\mathcal{M}^{GS}_{(r,c\ell,\chi)}$, E satisfies IT_0 (respectively WIT_0).*

Recall the definition of simple sheaves:

Definition 2.5.7. *If S is an abelian surface S over algebraic closed field k and E is a stable sheaf on S, then E is **simple** if $\mathrm{Hom}(E,E) = k$.*

The following is very useful and special to abelian surfaces:

Proposition 2.5.8. *If E is simple on an abelian surface then $\mathrm{ch}(E)$ satisfies Bogomolov inequality.*

Proof. Suppose E is a simple sheaf, then the Euler characteristic of E is given by

$$\chi(E,E) = \sum_{i=0}^{\dim S} (-1)^i \dim \mathrm{Ext}^i(E,E)$$

From Serre duality, since E is simple, then $\mathrm{Ext}^0(E,E) = \mathrm{Ext}^2(E,E[2]) = k$. Hence $\chi(E,E) = 2 - \dim \mathrm{Ext}^1(E,E)$. Mukai in [Muk84] showed that if E is simple then $\dim \mathrm{Ext}^1(E,E) \geq 2$. Then $-\chi(E,E) + 2 \geq 2$ and so $\chi(E,E) \leq 0$. Hence E satisfies Bogomolov inequality. □

The following example links moduli spaces to Fourier-Mukai transforms:

Example 2.5.9. *Let S be an abelian surface, $\boldsymbol{v}(E)$ isotropic Mukai vector,* $\dim \mathcal{M}_{\boldsymbol{v}}^{GS} = 2$, *then $\mathcal{M}_{\boldsymbol{v}}^{GS}$ is a fine moduli space of semi-homogeneous bundle and the universal sheaf $\mathbb{E}$ on $S \times \mathcal{M}_{\boldsymbol{v}}^{GS}$ gives rise to a Fourier-Mukai transform and all such a Fourier-Mukai transform arises in this way.*

Chapter 3

k-Very Ample Line Bundles

3.1 k-Very Ample Line Bundles

We begin this section with a brief review of k-very ampleness, see [Ter98a], [Ter98b], [BFS89] and [AB11] for more details. Throughout this section, let V be a smooth complete algebraic variety of dimension g over $\mathbb{C}$ and L an invertible sheaf on V. Denote by $\mathrm{Hilb}^k V$ the Hilbert scheme of length k, purely 0-dimensional subschemes of V.

Definition 3.1.1. *For each 0-dimensional subscheme X on V we can consider the restriction map*

ρ_X to X for the space of sections of L, which fits into the exact sequence:

$$0 \to H^0(V, L \otimes \mathcal{I}_X) \to H^0(V,L) \overset{\rho_X}{\to} H^0(\mathcal{O}_X) \to H^1(V, L \otimes \mathcal{I}_X) \to H^1(V,L) \to 0 \quad (3.1.1)$$

*L is called k-**very ample** if ρ_X is surjective for all purely 0-dimensional subscheme X of length $|X| \leq k+1$.*

In other words, if L is k-very ample then for each $X \in \mathrm{Hilb}^{k+1} V$ the sequence (3.1.1) associates a subspace of $H^0(V,L)$ of codimension $k+1$ and this map yields a morphism

$$\varphi_k : \mathrm{Hilb}^{k+1} V \to \mathrm{Grass}(k+1, \Gamma(L)) \quad (3.1.2)$$

where Grass is the Grassmannian of all $k+1$ quotients of $\Gamma(L)$. This map takes every $X \in \mathrm{Hilb}^{k+1} V$ into the quotient $\Gamma(L) \to \Gamma(L \otimes \mathcal{O}_X)$. In particular, L is $k+1$-very ample if and only if φ_k is embedding (see [CG90]).

Remark 3.1.2. *The following follows easily from the definition 3.1.1*

- *L is 0-very ample if and only if L is generated by global section.*
- *L is 1-very ample if and only if it is very ample.*

Moreover, there is a relation between k-very ampleness and "weak index theorem" which was used in [PP03] and we will prove this relation in the following proposition:

Proposition 3.1.3. *Let (S, Φ) be an abelian variety and L satisfied IT_0. The following are equivalent:*

1. *L is k-very ample.*
2. *$L \otimes \mathcal{I}_X$ is WIT_0 (and hence IT_0) for all $X \in \mathrm{Hilb}^{k+1} S$.*

Proof. "(1.) $\Rightarrow$ (2.)" Suppose that L is k-very ample and $L \otimes \mathcal{I}_X$ is not WIT_0 for some purely 0-dimensional subscheme X of length $|X| \leq k+1$. Then there exists $\hat{x} \in \hat{S}$ such that $H^1(L\mathcal{P}_{\hat{x}}\mathcal{I}_X) \neq 0$. Let $\psi_L(x) = P_{\hat{x}}$, then $H^1(\tau^*_{-x}(L)\mathcal{I}_X) \neq 0$ and so $H^1(\tau^*_{-x}(L \otimes \mathcal{I}_{\tau_x X})) \neq 0$. Hence $H^1(L \otimes \mathcal{I}_{\tau_{-x} X}) \neq 0$ where $|\tau_{-x} X| \leq k+1$ and this contradicts the assumption.

"(2.) $\Rightarrow$ (1.)" " Now suppose that $L \otimes \mathcal{I}_X$ is WIT_0 for all purely 0-dimensional subscheme X of length $|X| \leq k+1$ and we want to show that k-very ample. Since $L \otimes \mathcal{I}_X$ satisfies WIT_0, then $H^1(L \otimes \mathcal{I}_X) = 0$. Hence L is k-very ample by definition. □

In other words, the previous Proposition shows that L is k-very ample if and only if $\mathcal{M}^{GS}_{(1,\ell,\frac{1}{2}\ell^2-k-1)}$ is IT_0 (see Definition 1.5.13).

From the definition 3.1.1 , we can easily observe that if L is k-very ample then L is $(k-1)$-very ample. So it makes sense to consider the critical value of k which is the least non negative integer such that L is not $(k+1)$-very ample and denote it by $\phi(L)$.

Definition 3.1.4. *Let* $\mathrm{Amp}(S)$ *be the ample cone of* S. *Define a map*

$$\phi : \mathrm{Amp}(S) \to \mathbb{Z}_{\geq -1}$$

which takes L *into the integer* k *such that* L *is* k*-very ample but not* $(k+1)$*-very ample. Define* $\phi_L(n) := \phi(L^n)$, *and* $\phi(n)$ *if* L *is understood.*

Example 3.1.5. *Let* (V, L) *be a principally polarized abelian variety. Then* $\phi_L(2) = 0$ *(see [PP04], §.1).*

A map $\phi(n)$ is bounded and upper and lower bound are known. As L is an ample bundle then it stisfies IT_0 which means that $H^1(L) = 0$. Hence an upper bound of $\phi(n)$ can be the Euler characteristic of $L^n \otimes \mathcal{I}_X$ which is $\chi(L^n \otimes \mathcal{I}_X) = \chi(L^n) - |X|$. Thus $\phi(n)$ is bounded above by $\chi(L^n) - 1$.

On the other hand, Terakawa in [Ter98b] gave necessary and sufficient conditions for line bundle to be k-very ample for abelian surfaces in the following theorem

Theorem 3.1.6. *[Ter98b, Theorem 1.1] Let* S *be an abelian surface,* L *an ample line bundle on* S *and* $k \in \mathbb{Z}_{\geq 0}$ *then the following are equivalent:*

- *L is k-very ample;*
- *$c_1^2(L) \geq 4k+6$ and there is no effective divisor D satisfying the following*

$$2\sqrt{(2k+3)(p_a(D)-1)} \leq L \cdot D \leq 2p_a(D) + k - 1 \leq 2k+1,$$

  where $p_a(D) = 1 - \chi(\mathcal{O}_D)$ is the arithmetic genus of D.

Hence $c_1^2(L^n) \geq 4\phi(n) + 6$ and so $2n^2 d - 6 \geq 4\phi(n)$. Thus $\frac{1}{2}(n^2 d - 3)$ is an lower bound of $\phi(n)$. On the other side, Reider's theorem can give us an lower bound of $\phi(n)$ (see [BFS89] or [AB11, §2]).

Theorem 3.1.7. *[AB11, Reider's Theorem] If* S *is a projective smooth surface and* L *is an ample line bundle on* S *such that* $c_1^2(L^n) > (k+2)^2$, *then* $H^1(S, L^n \otimes \mathcal{I}_X) = 0$ *for all 0-subscheme* X *of length* $|X| \leq k+1$.

From Reider's theorem and Proposition 3.1.3, if L^n is k-very ample, then $c_1^2(L^n) > (k+2)^2$. Then $2dn^2 \leq (\phi(n)+2)^2$ and so $\sqrt{2d}n - 2 \leq \phi(n)$. Hence $\lceil \sqrt{2d}n \rceil - 3$ is a upper bound of $\phi(n)$, but it is not sharp even for $d = 1$ and $n = 2$.

3.2 k-Very Ample Line Bundles on an abelian surface and $\rho = 1$

Throughout this section, let S be an abelian surface, L an ample line bundle on S and $\mathrm{NS}(S) = \mathbb{Z}c_1(L)$. We have shown in a previous section that the lower and upper bound for ϕ are known, but in this section we will give the critical value of $\phi(n)$.

3.2.1 Principal polarization Line Bundle

Let (S, L) be a principally polarized abelian surface. Firstly, we recall the homomorphism ψ_L associated to line bundle L where $\psi_L : S \to \hat{S}$ takes x into $\tau_x L \otimes L^{-1}$. We will give the upper bound of $\phi(n)$ for $n > 1$, but we need first to recall the definition of collinear 0-dimensional

Definition 3.2.1. *If X is a 0-dimensional, then we say that X is **collinear** if X is a subset of a translation of the polarization divisor D_L, i.e. $X \subset \tau_x D_L$ for some $x \in S$ and $H^0(L \otimes \mathcal{P}_{-\psi_L(x)} \otimes \mathcal{I}_X) \neq 0$.*

Proposition 3.2.2. *Let (S, L) be an irreducible principally polarized abelian surface, then L^n is not $(2n-3)$-very ample.*

Proof. Let X be a 0-dimensional collinear subscheme of D_L of length $2(n-1)$. Then there exists a non-zero map

$$\mathcal{O} \to L \otimes \mathcal{P}_{-\psi_L(x)} \otimes \mathcal{I}_X \tag{3.2.1}$$

Twist (3.2.1) by L^{n-1}, we get a sequence

$$0 \to L^{n-1} \to L^n \otimes \mathcal{I}_X \to Q \to 0$$

where Q is quotient with Chern character $\mathrm{ch}(Q) = (0, \ell, (2n-1) - |X|)$. Now suppose that Q is IT_0. Since the transform of Q, $\hat{Q}$ has the Chern character $\mathrm{ch}(\hat{Q}) = ((2n-1) - |X|, -\ell, 0) = (1, -\ell, 0)$, but $\hat{Q}$ is locally-free which is impossible. So Q is not IT_0 and then $L^n \otimes \mathcal{I}_X$ is not IT_0. □

Corollary 3.2.3. *Let (S, L) be an irreducible principally polarized abelian surface, then $\phi(n) \leq 2n - 4$ for $n \geq 2$.*

To prove that the lower bound is sharp, we induct on n and the following lemmas prove that this bound is sharp for the base case when $n = 3$. We need first to prove the following technical lemmas

Lemma 3.2.4. *If E is a pure torsion sheaf on S with $\mathrm{ch}(E) = (0, \ell, -1)$ then $E \cong \mathcal{P}_{\hat{x}}/(L^{-1} \otimes \mathcal{P}_{\hat{y}})$ for some $\hat{y}, \hat{x} \in \hat{S}$.*

Proof. The Fourier-Mukai transform of E has Chern character $(-1, -\ell, 0)$ and so if E were IT_1 then its transform would be a line bundle of Chern character $(1, \ell, 0)$ which is impossible. So E is not IT_0. Hence, $H^0(E \otimes \mathcal{P}_{-\hat{x}}) \neq 0$ for some $\hat{x}$. Then we have a non-trivial map $\mathcal{P}_{\hat{x}} \to E$. Since E is pure, the Syzygy Theorem implies that the kernel is locally-free. From the Chern character it takes the form $L^{-1} \otimes \mathcal{P}_{\hat{y}}$ for some $\hat{y} \in \hat{S}$. Then the Chern character also tells us that the map must surject. □

Lemma 3.2.5. *If E is a pure torsion subsheaf of X with $\mathrm{ch}(E) = (0, \ell, r)$.*

- *If $r < 1$, then E satisfies WIT_1.*
- *If $r = 1$, then E does not satisfy IT_0 nor IT_1.*
- *If $r \geq 2$, then E satisfies IT_0.*

Proof. By the previous lemma, when $r = -1$, we have an exact sequence

$$0 \to L^{-1} \otimes \mathcal{P}_{\hat{y}} \to \mathcal{P}_{\hat{x}} \to E \to 0 \tag{3.2.2}$$

Applying Φ to (3.2.2), we get a long exact sequence

$$0 \to \Phi^1(E) \to \Phi^2(L^{-1} \otimes \mathcal{P}_{\hat{y}}) \xrightarrow{\gamma} \Phi^2(\mathcal{P}_{\hat{x}}) \to \Phi^2(E) \to 0$$

Since E is supported in dimension one, then $\Phi^2(E) = 0$. Also, $\Phi^0(L^{-1} \otimes \mathcal{P}_{\hat{y}}) = \Phi^0(P_{\hat{x}}) = 0$ and so $\Phi^0(E) = 0$. Hence E is IT_1 and note that $\Phi^1(E)$ is torsion free.

Now suppose E is a pure torsion sheaf of Chern character $(0, \ell, 0)$. Pick any $x \in \operatorname{supp}(E)$ and a non-zero map $E \to \mathcal{O}_x$. Then the kernel F is pure of Chern character $(0, \ell, -1)$ and so is IT_1 with torsion-free transform. We also have a non-split short exact sequence:

$$0 \to F \to E \to \mathcal{O}_x \to 0 \tag{3.2.3}$$

giving,

$$0 \to \Phi^0(E) \to \Phi^0(\mathcal{O}_x) \xrightarrow{\delta} \Phi^1(F) \to \Phi^1(E) \to 0. \tag{3.2.4}$$

Since $\Phi^0(\mathcal{O}_x) = \mathcal{P}_x$ and $r(\Phi^1(F)) = 1$, Lemma 1.5.8 implies that $\delta \neq 0$ and so it must inject so $\Phi^0(E) = 0$ and $\Phi^1(E)$ is a torsion sheaf.

Repeating this construction, with $\operatorname{ch}(E) = (0, \ell, 1)$, we get (3.2.4) where $\Phi^1(F)$ is torsion. If E is IT_0, then $\Phi^0(E)$ is locally-free of Chern character $(1, -\ell, 0)$ which is impossible. But $r(\Phi^0(E)) = 1$ and $c_1(\Phi^0(E)) = \ell$ so $\Phi^1(E)$ is a skyscraper, with $\chi(\Phi^1(E)) = 1$.

The first part follows because if E has Chern character $(0, \ell, 2k-1)$ for some integer $k < 0$. Then $E \otimes L^k$ is pure and torsion of Chern character $(0, \ell, -1)$. Then $H^0(E \otimes \mathcal{P}_{\hat{x}}) \cong H^0((E \otimes L^k) \otimes L^{-k} \otimes \mathcal{P}_{\hat{x}}) = 0$ by lemma 1.5.8 for all $\hat{x} \in \hat{X}$. So part 1 follows for all odd negative r. Then the construction above shows that it remains true for even r because $\gamma : \Phi^0(\mathcal{O}_x) \to \Phi^1(E)$ must be non-zero and so injects as $\Phi^0(\mathcal{O}_x)$ is locally-free of rank 1.

Repeating with $\mathrm{ch}(E) = (0, \ell, 2)$, applying Φ to sequence (3.2.3) where $\mathrm{ch}(F) = (0, \ell, 1)$, then we get

$$0 \to L^{-1} \to \Phi^0(E) \to \mathcal{P}_{\hat{x}} \xrightarrow{\delta} \mathcal{O}_{\hat{x}} \to \Phi^1(E) \to 0$$

By Lemma 1.5.8, $\delta \neq 0$ and then it is surjective. Hence $\Phi^1(E) = 0$.

Assume that E satisfies IT_0 for $r = m$. If $r = m+1$, F satisfies IT_0 by induction. Since $\mathcal{O}_x$ satisfies IT_0, then E is IT_0 for $r = m+1$ and the statement is true for all $r \geq 2$.

□

Lemma 3.2.6. *Let L be a principal polarization line bundle, then L^3 is 2-very ample.*

Proof. We start from a classical fact that $\Phi^0(L^n)$ is μ-stable for all $n > 0$ (See [Muk78], Proposition 3.16- p351). Let Y be a 0-subscheme of length 3. Apply Φ^* to the twisted structure sequence of $\mathcal{I}_Y$

$$0 \to L^3\mathcal{I}_Y \to L^3 \to \mathcal{O}_Y \to 0$$

we get this long exact sequence

$$0 \to \Phi^0(L^3\mathcal{I}_Y) \to \Phi^0(L^3) \to \Phi^0(\mathcal{O}_Y) \to \Phi^1(L^3\mathcal{I}_Y) \to 0$$

Let $\mathrm{ch}(\Phi^0(L^3\mathcal{I}_Y)) = (6, \alpha\ell, \chi)$ and $\mathrm{ch}(\Phi^1(L^3\mathcal{I}_Y)) = (0, a\ell, b)$, $a \geq 0$. Since $\Phi^0(L^3)$ is μ-stable, then $\mu(\Phi^0(L^3\mathcal{I}_Y)) = \frac{\alpha.\ell^2}{6} < \mu(\Phi^0(L^3)) = \frac{-2}{3}$. Hence $\alpha < -2$ and $a = \alpha + 3$. Therefore $a < 1$ and so $a = 0$. Also, $\chi = b + 1$. Assume that $b \neq 0$ then $\chi \geq 2$.

claim 3.2.7. *$\Phi^0(L^3\mathcal{I}_Y)$ is μ-semistable.*

Proof. Let E be a destabilizing subsheaf of $\Phi^0(L^3\mathcal{I}_Y)$ with Chern character $\mathrm{ch}(E) = (r, \beta.\ell, \chi(E))$, then E injects into $\Phi^0(L^3)$ and we have a short exact sequence

$$0 \to E \to \Phi^0(L^3\mathcal{I}_Y) \to Q \to 0$$

where Q is a quotient sheaf with $\mathrm{ch}(Q) = (1, -\ell, b+1-\chi)$. Since $\Phi^0(L^3)$ is μ-stable, then

$$\mu(L^3) > \mu(E) \geq \mu(\Phi^0(L^3\mathcal{I}_Y)).$$

Therefore

$$\frac{-2}{3} > \frac{2\beta}{r} > -1. \tag{3.2.5}$$

Since $1 \leq r < 6$, then $\frac{-1}{3} > \beta > -3$. Thus $\beta = -1$ or -2. If $\beta = -1$, then there is no solutions of (3.2.5). Hence $\beta = -2$, then $6 > r > 4$ so $r = 5$. Without lost of generality, assume that Q is torsion free sheaf which means $\chi(E) \geq b \geq 0$. By the Bomogolov inequality, $\chi(E) < \frac{4}{5}$. Then the only possible destabilizer sheaf has the Chern character $\mathrm{ch}(E) = (5, -2\ell, 0)$. Since $\mathrm{ch}(E)$ is the only possible Chern character of a destabilizer sheaf, Q must be torsion-free. Hence $Q = L^{-1}$ up to twist and it is WIT_2. Also $\Phi^0(L^3\mathcal{I}_Y)$ is WIT_2, then E is WIT_2.

Applying the Mukai spectral sequence to $L^3\mathcal{I}_Y$ to get a long exact sequence

$$0 \to \hat{\Phi}^0\Phi^1(L^3\mathcal{I}_Y) \to \hat{\Phi}^2\Phi^0(L^3\mathcal{I}_Y) \to L^3\mathcal{I}_Y \to \hat{\Phi}^1\Phi^1(L^3\mathcal{I}_Y) \to 0 \tag{3.2.6}$$

The Fourier-Mukai transform $\hat{E}$ of E has rank zero, then it is torsion and maps to $\hat{\Phi}^2\Phi^0(L^3\mathcal{I}_Y)$. Then we have the following diagram from sequence (3.2.6)

$$\begin{tikzcd}
\hat{\Phi}^0\Phi^1(L^3\mathcal{I}_Y) \arrow[r] & \hat{\Phi}^2\Phi^0(L^3\mathcal{I}_Y) \arrow[r] & L^3\mathcal{I}_Y \\
 & \hat{E} \arrow[u, "\delta"] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

Since $\hat{E}$ is a torsion sheaf and $L^3\mathcal{I}_Y$ is a torsion free, then the composition γ is zero. On the other hand, $\hat{\Phi}^0\Phi^1(L^3\mathcal{I}_Y)$ is locally free and so $\delta = 0$. Hence there is no destabilizer sheaf of $\Phi^0(L^3\mathcal{I}_Y)$. $\square$

Returning to the proof, the claim and Bomogolov inequality imply that $\chi \leq \frac{3}{2}$ which contradicts our assumption. Hence $b = 0$ and then $L^3\mathcal{I}_Y$ is WIT_0 and hence L^3

is 2-very ample. □

Lemma 3.2.8. *Let L be a principal polarization line bundle, then $\phi(3) = 2$.*

Proof. Now pick a 0-subscheme $Z \subset D_L$ with $|Z| = 4$. Then there is a short exact sequence

$$0 \to L^2 \to L^3\mathcal{I}_Z \to Q \to 0,$$

where Q is a pure torsion sheaf with $\mathrm{ch}(Q) = (0, \ell, 1)$. By Lemma 3.2.5, Q is neither IT_0 nor IT_1, then $L^3\mathcal{I}_Z$ is not IT_0. Hence $L^3\mathcal{I}_Z$ is not WIT and then L^3 is not 3-very ample. □

Lemma 3.2.9. *$L^n\mathcal{I}_{X'}$ is IT_0 for any 0-dimensional X' with $|X'| = 2n-3$ and $n \geq 3$.*

Proof. We will use the induction on n to prove this lemma. If $n = 3$, L^3 is 2-very ample (Lemma 3.2.8). Suppose that the statement is true for $n > 2$, i.e. $L^n\mathcal{I}_{X'}$ is IT_0 where $|X'| = 2n-3$.

Now given X' be a 0-dimensional with $|X'| = 2n-1$ and $X'' \subset X'$ such that $|X' \setminus X''| = 2$, then there exists a short exact sequence

$$0 \to \mathcal{P}_{\hat{x}} \otimes L^n\mathcal{I}_{X''} \to L^{n+1}\mathcal{I}_{X'} \to T \to 0, \tag{3.2.7}$$

for some $\hat{x} \in \hat{S}$. This is because scheme theoretic support of T is $X'' \setminus X'$ must be collinear (See [Mac12]). From the assumption, $L^n\mathcal{I}_{X''}$ is IT_0 and T has Chern character $\mathrm{ch}(T) = (0, \ell, 2n-1)$.

To prove that T is pure, let A be a 0-dimensional subscheme of T with Chern character $\mathrm{ch}(A) = (0, 0, \alpha)$ where $\alpha \geq 0$, then from a sequence (3.2.7) we get this diagram:

$$\begin{tikzcd}
& & B \\
\mathcal{P}_x \otimes L^n\mathcal{I}_{X''} \arrow[r] \arrow[d] & L^{n+1}\mathcal{I}_{X'} \arrow[r] \arrow[ur] & T \arrow[u] \\
D \arrow[ur] \arrow[rr] & & A \arrow[u]
\end{tikzcd}$$

Thus D is a torsion free with $\mathrm{ch}(D) = (1, n\ell, (n-1)^2 + 2 + \alpha)$, then $0 \le \alpha \le 2n-3$ and so $\chi(B) \ge 2$. Lemma 3.2.5 implies that B is IT_0 and then T is IT_0. Hence, $L^{n+1}\mathcal{I}_{X'}$ is IT_0.

□

The following is the result of the previous Lemma:

Corollary 3.2.10. *$\phi(L^n) = 2n-4$ for $n \ge 3$.*

3.2.2 Polarization Line Bundle of Degree d

In this section we will generalize the value of $\phi(n)$ for line bundle L of degree d and we start with the exceptional case when $n = 1$. Bauer and Szemberg, in [BS97], proved indirectly the case $n = 1$ for primitive Line bundle in the following propositions

Proposition 3.2.11. *Let S be an abelian surface with Picard number 1, let L be an ample line bundle of type $(1, d)$, $d \ge 1$, and let k be a nonnegative integer. If $d \ge 2k+3$, then L is k-very ample.*

Proof. See [BS97, Prop 3.2]. □

From Proposition 3.2.11, $d \ge 2\phi_L(1) + 3$ then $d - 3 \ge 2\phi_L(1)$. On the other hand, applying Proposition 5.3.2 we get $d - 32 \le 2\phi_L(1)$. And so it follows that if L is an ample line bundle of degree $d \ge 1$ on an abelian surface X with Picard rank 1, then $\phi_L(1) = \left\lfloor \dfrac{d-3}{2} \right\rfloor$.

We give an alternative direct proof using the Bogomolov inequality:

Proposition 3.2.12. *Let S be an abelian surface with $\mathrm{NS}(S) = \langle \ell \rangle$. If L is an ample line bundle with $c_1(L)^2 = \ell^2 = 2d$ on an abelian surface S, then*

$$\phi_L(1) = \left\lfloor \frac{d-3}{2} \right\rfloor$$

Proof. The Chern character of $E = L \otimes \mathcal{I}_X$ is $(1, \ell, d - |X|)$. Let $F = \Phi(E)$, then $\mathrm{ch}(F) = (d - |X|, -\ell, 1)$. If F is stable, then F^{**} is stable and there is a short exact sequence

$$o \to F \to F^{**} \to \mathcal{O}_W \to 0$$

If F is locally-free then W is empty. In other words, F is locally-free when there are no stable sheaves with Chern character $(d - |X|, -\ell, 2)$. These exist exactly when the Bogomolov inequality does not hold for such a Chern character. This gives us the criterion $2(d - |X|) > d$, so $|X| \leq \left\lceil \frac{d}{2} \right\rceil = \left\lfloor \frac{d-1}{2} \right\rfloor$. Hence $\mathcal{M}^{\mathrm{GS}}_{(1,\ell,d-|X|)}$ is IT_0 if and only if $|X| \leq \left\lfloor \frac{d-1}{2} \right\rfloor$. Then $\phi_L(1) = \left\lfloor \frac{d-3}{2} \right\rfloor$. $\square$

Chapter 4

Moduli Space of Sheaves on Products of Elliptic Curves

Let (S,ℓ) be a principal polarized abelian surface such that S is a product of two elliptic curves $\mathcal{C}_1$ and $\mathcal{C}_2$, and L an ample line bundle on S where $L = L_1 \otimes L_2$ with $c_1(L) = n_1\ell_1 + n_2\ell_2 = (n_1, n_2)$ where $\ell_i = c_1(L_i)$. We do not assume $\rho(S) = 2$ but when $\rho(S) > 2$ we restrict our attention to such first Chern classes.

4.1 Moduli Space of Sheaves with $c_1 = (\pm 1, 0)$

For any line bundle M on S, recall, $\varphi_M : S \to \hat{S}$ is defined by $\varphi_M(x) = \tau_x^* M \otimes M^*$ is an isogeny and $K(M) := \operatorname{Ker}\varphi_M$. If M is an ample then $K(M)$ is finite. For $M = L_1$, $\varphi_{L_1} : \mathcal{C}_1 \times \mathcal{C}_2 \to \hat{S}$ has image $\hat{\mathcal{C}}_2$ and kernel $\mathcal{C}_1$. If $x \in S$, then $x = x_1 + x_2$ for unique $x_i \in \mathcal{C}_i$, $i = 1,2$.

Lemma 4.1.1. *If M is a line bundle with $c_1(M) = (-1,0)$, then $M^* \cong \tau_x^* L_1 \otimes \mathcal{P}_{\hat{x}}$ for some $x \in \mathcal{C}_2$, $\hat{x} \in \hat{\mathcal{C}}_1$ and* $h^0(M^* \otimes \mathcal{P}_{\hat{y}}) = \begin{cases} 1 \text{ if } \hat{y}_1 = -\hat{x} \\ 0 \text{ otherwise} \end{cases}$.

Proof. Adapted from ([Mum74], vi, p. 75), if M is a line bundle with $c_1(M) = (-1,0)$, then $M \cong L_1^* \otimes \mathcal{P}_{\hat{x}}$. For each $x_i \in \mathcal{C}_i$, $\tau_{x_i}^* L_i \cong L_i$, $i = 1,2$ and $\tau_{x_1}^* L_2 \cong L_2 \otimes \mathcal{P}_{\varphi_{L_2}(x_1)}$,

$\tau^*_{x_2} L_1 \cong L_1 \otimes \mathcal{P}_{\varphi_{L_1}(x_2)}$. Then $K(L_i) \cong \mathcal{C}_i$. Hence

$$M \cong L_1^* \otimes \mathcal{P}_{\hat{x}} \cong L_1^* \otimes \mathcal{P}_{\hat{x}_1} \otimes \mathcal{P}_{\hat{x}_2} \cong \tau^*_{\varphi^{-1}_{\hat{L}_1(\hat{x}_2)}} L_1^* \otimes \mathcal{P}_{\hat{x}_1} \cong \tau^*_{x_2} L_1^* \otimes \mathcal{P}_{\hat{x}_1}$$

where $x_2 := \varphi^{-1}_{L_1}(\hat{x}_2)$. On the other hand, to find $h^0(L_1 \times \mathcal{P}_{\hat{y}})$ we will use the Künneth formula. If $\hat{y}_1 \neq 0$, then

$$H^0(\mathcal{C}_1 \times \mathcal{C}_2; \pi_2^* \tilde{L}_2 \otimes \pi_1^* \mathcal{P}_{\hat{y}_1} \otimes \pi_2^* \mathcal{P}_{\hat{y}_2}) \cong H^0(\mathcal{C}_1; \mathcal{P}_{\hat{y}_1}) \otimes H^0(\mathcal{C}_2; \tilde{L}_2 \otimes \mathcal{P}_{\hat{y}_2}) = 0$$

and $\tilde{L}_2$ is principal polarization of $\mathcal{C}_2$.

□

Corollary 4.1.2. *The moduli space $\mathcal{M}_{(1,(-1,0),0)}$ of sheaves with Chern character $(1,(-1,0),0)$ is fine and $\mathcal{M}_{(1,(-1,0),0)} \cong \hat{\mathcal{C}}_1 \times \mathcal{C}_2$.*

Proof. Let $f : \mathcal{C}_1 \times \mathcal{C}_2 \times \hat{\mathcal{C}}_1 \times \mathcal{C}_2 \to S \times \hat{S}$ be the isomorphism $1 \times 1 \times 1 \times \varphi_{L_1}|_{\mathcal{C}_2}$ and define $\mathbb{E} = \pi_1^* \tilde{L}_1 \otimes f^* \mathcal{P}$, where $\pi_1 : \mathcal{C}_1 \times \mathcal{C}_2 \times \hat{\mathcal{C}}_1 \times \mathcal{C}_2 \to \mathcal{C}_1$ and $\mathcal{P}$ is Poincaré bundle. The restriction of $\mathbb{E}$ at $(\hat{x}_1, x_2)$ is the line bundle $\mathbb{E}|_{(\hat{x}_1, x_2)} = \tau^*_{x_2} L_1^* \otimes \mathcal{P}_{\hat{x}_1}$, then $\mathcal{M}_{(1,(-1,0),0)} \cong \hat{\mathcal{C}}_1 \times \mathcal{C}_2$ and $\mathbb{E} \to S \times \mathcal{M}_{(1,(-1,0),0)}$ is the universal sheaf parametrizing torsion free sheaves of Chern character $(1,(-1,0),0)$ by Lemma 4.1.1.

□

Note 4.1.3. *If $c_1(M) = (-1,0)$ then for any $y \in S$,*

$$\begin{aligned}
\varphi_{M^*}(y) &= \varphi_{\tau^*_x L_1}(y) \\
&= \tau^*_y \tau^*_x L_1 \otimes (\tau^*_x L_1)^* \\
&= \tau^*_x(\tau^*_y L_1 \otimes L_1^*) \\
&= \tau^*_y L_1 \otimes L_1^* = \varphi_{L_1}(y).
\end{aligned}$$

i.e. $\varphi_{M^*} = \varphi_{L_1}$.

Corollary 4.1.4. $\Phi^2(L_1^{-1}) \cong \mathcal{O}_{\hat{\mathcal{C}}_2} \otimes \hat{L}_1$

Proof. The Chern character of $\mathrm{ch}(\Phi^2(L_1^{-1})) = (0,(0,1),1)$, then $\Phi^2(L_1^{-1}) \cong \mathcal{O}_{\hat{\mathcal{C}}_2} \otimes \hat{L}_1 \otimes \mathcal{P}_{x_1}$ for some $x \in \mathcal{C}_1$ and by Lemma (4.1.1) $\Phi^2(L_1^{-1})$ is supported on $\hat{\mathcal{C}}_2$. But $\hat{\mathcal{O}}_{\mathcal{C}_1} \cong \mathcal{O}_{\hat{\mathcal{C}}_2}$ since $H^0(\hat{\mathcal{O}}_{\mathcal{C}_1}) \neq 0$ and so from the short exact sequence $0 \to L^{-1} \to \mathcal{O} \to \mathcal{O}_{\mathcal{C}_1} \to 0$ we have $\Phi^2(L_1^{-1}) \cong \mathcal{O}_{\hat{\mathcal{C}}_2} \otimes \hat{L}_1$. □

Lemma 4.1.5. *If T is a pure torsion sheaf with* $\mathrm{ch}(T) = (0,(1,0),0)$, *then there exists an $x \in \mathcal{C}_2$ and $\hat{y} \in \hat{\mathcal{C}}_1$ and a short exact sequence*

$$0 \to (\tau_x^* L_1)^{-1} \to \mathcal{P}_{\hat{y}} \to T \to 0$$

Proof. Consider a pure torsion sheaf T with Chern character $(0,(1,0),0)$. Then T is supported on some translate of $D_{\mathcal{C}_1}$ and so there exists $\hat{y} \in \hat{\mathcal{C}}_1$ and $x \in \mathcal{C}_2$ such that $T \cong \mathcal{O}_{\mathcal{C}} \otimes \mathcal{P}_{\hat{y}}$ where $\mathcal{C} = \tau_x \mathcal{C}_1$. But there exists a short exact sequence

$$0 \to L_1^{-1} \to \mathcal{O} \to \mathcal{O}_{\mathcal{C}} \to 0 \tag{4.1.1}$$

Applying τ_x^* to (4.1.1), we get the exact sequence

$$0 \to \tau_x^* L_1^{-1} \to \mathcal{O} \to \mathcal{O}_{\mathcal{C}} \to 0 \tag{4.1.2}$$

Applying $\_ \otimes \mathcal{P}_{\hat{y}}$ to (4.1.2), we get the exact sequence

$$0 \to \tau_x^* L_1^{-1} \otimes \mathcal{P}_{\hat{y}} \to \mathcal{P}_{\hat{y}} \to \mathcal{O}_{\mathcal{C}} \otimes \mathcal{P}_{\hat{y}} \to 0 \tag{4.1.3}$$

Hence

$$0 \to \tau_x^* L_1^{-1} \otimes \mathcal{P}_{\hat{y}} \to \mathcal{P}_{\hat{y}} \to T \to 0 \tag{4.1.4}$$

is exact. □

Lemma 4.1.6. *Let $S = \mathcal{C}_1 \times \mathcal{C}_2$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are elliptic curves, and T a stable pure torsion subsheaf of S with* $\mathrm{ch}(T) = (0,(1,0),\chi)$, *then*

- *if $\chi = 0$, then T is* WIT_1 *and $\hat{T}$ is pure and $\hat{T} \in$* $\mathrm{Coh}^{\leq 1}$,

- *if* $\chi < 0$, *then* T *is* IT_1,

- *if* $\chi \geq 1$, *then* T *is* IT_0.

Proof. Consider T a pure torsion sheaf with Chern character $\mathrm{ch}(T) = (0,(1,0),0)$. Then by Lemma (4.1.5) we have a short exact sequence

$$0 \to \tau_x^* L_1^{-1} \otimes \mathcal{P}_{\hat{y}} \to \mathcal{P}_{\hat{y}} \to T \to 0 \tag{4.1.5}$$

Since L_1^{-1} and $\mathcal{P}_{\hat{y}}$ satisfy WIT_2, then T is WIT_1

$$0 \to \hat{T} \to \Phi^2(\tau_x^* L_1^{-1} \otimes \mathcal{P}_{\hat{y}}) \to \mathcal{O}_{-\hat{y}} \to 0 \tag{4.1.6}$$

Since $\Phi(\tau_x^* -) \cong \Phi(-) \otimes \mathcal{P}_{-x}$ ([Muk81]), then $\Phi^2(\tau_x^* L_1^{-1} \otimes \mathcal{P}_{\hat{y}}) = \tau_{\hat{y}}^* \Phi^2(L_1^{-1}) \otimes \mathcal{P}_{-x} = \mathcal{O}_{\tau_{-\hat{y}}\hat{C}_2} \otimes \hat{L}_1 \otimes \mathcal{P}_{-x}$, by Corollary 4.1.4.

Dekker in [Dek97] gave a proof of the statement for $\chi < 0$ in (Theorem 4.9-p. 45) but we will reprove it here by using induction. Suppose first that $\mathrm{ch}(T) = (0,(1,0),0)$. Pick $x \in \mathrm{Supp}(T)$ and then there exists a non-zero map $T \to \mathcal{O}_x$ with kernel K. Then we have a short exact sequence:

$$0 \to K \to T \to \mathcal{O}_x \to 0 \tag{4.1.7}$$

where $\mathrm{ch}(K) = (0,(1,0),-1)$. Applying Φ to (4.1.7), we obtain a long exact sequence

$$0 \to \mathcal{P}_x \to \Phi^1(K) \to \Phi^1(T) \to \mathcal{O} \to 0 \tag{4.1.8}$$

By hypothesis T is WIT_1 and $\Phi^1(T)$ is pure, then $\Phi^0(K) = 0$. Hence K is WIT_1. To prove that $\Phi^1(K)$ is torsion free, let A be the torsion subsheaf of $\Phi^1(K)$ supported on

$\mathcal{C}_2$with $\mathrm{ch}(A) = (0,(0,1),\alpha)$ where $\alpha \le 0$ and we have the following diagram

$$\begin{tikzcd}
 & & Q \arrow[r] & \mathcal{O}_Z & \\
0 \arrow[r] & \mathcal{P}_x \arrow[r] \arrow[ur] & \Phi^1(K) \arrow[r] \arrow[u] & \Phi^1(F) \arrow[r] \arrow[u] & 0 \\
 & & A \arrow[u] \arrow[ur, hook] & &
\end{tikzcd}$$

where Q is the quotient sheaf with $\mathrm{ch}(Q) = (1,(0,0),-\alpha)$ which is an ideal sheaf $\mathcal{I}_{Z'} \otimes \mathcal{P}_x$ for Z' with length $|Z'| = \alpha$. But $H^0(\mathcal{I}_{Z'} \otimes \mathcal{P}_x) = 0$ unless $|Z'| = 0$ then $\alpha = 0$ and so $\mathcal{O}_Z = 0$. Hence $A \cong \Phi^1(F)$ which splits the sequence and that is a contradiction. Therefore if $\chi = -1$, then $\Phi^1(K)$ is torsion free with $\mathrm{ch}(\Phi^1(K)) = (1,(0,1),0)$ which is L_2 up to twist and then satisfies WIT_1. If $\chi < -1$, then $\Phi^1(F)$ is locally-free and so $\Phi^1(K)$ is locally-free as well.

We now turn to the $\chi > 0$ cases. Given F a pure torsion sheaf with $\mathrm{ch}(F) = (0,(1,0),0)$. As above, we have a short exact sequence:

$$0 \to K \to F \to \mathcal{O}_x \to 0 \tag{4.1.9}$$

for any $x \in \mathrm{Supp}(F)$. Applying $\mathrm{Ext}^*(\mathcal{O}_x, \_)$ to(4.1.9), we get an exact sequence

$$\mathrm{Ext}^1(\mathcal{O}_x, F) \to \mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Ext}^2(\mathcal{O}_x, K) \to 0$$

Since $\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \cong \mathbb{C}^2$ and $\mathrm{Ext}^2(\mathcal{O}_x, K) = \mathrm{Hom}(K, \mathcal{O}_x)^* = \mathbb{C}$ by Serre duality. Then $\mathrm{Ext}^1(\mathcal{O}_x, F) \neq 0$ and there exists a non-split short exact sequence

$$0 \to F \to T \to \mathcal{O}_x \to 0 \tag{4.1.10}$$

Applying Φ to (4.1.10) , then we get

$$0 \to \Phi^0(T) \to \mathcal{P}_x \xrightarrow{\delta} \Phi^1(F) \to \Phi^1(T) \to 0. \tag{4.1.11}$$

Lemma 1.5.8 implies that $\delta \neq 0$ and for contradiction suppose it is not surjective and its image is supported on curve and so $\Phi^1(T) = \mathcal{O}_Z$ for some 0-dimensional Z with length $|Z| = \alpha$. But $\Phi^1(F)$ is locally-free, then $\alpha = 0$. Hence $\Phi^1(T) = 0$ and then T is IT_0. Now suppose T' with Chern character $\mathrm{ch}(T') = (0,(1,0),\chi)$ for $\chi \geq 1$ is IT_0, then we have a short exact sequence

$$0 \to T' \to T \to \mathcal{O}_x \to 0$$

By the induction hypothesis T is IT_0 for all $\chi \geq 1$. □

4.2 k-Very Ample Line bundles on $\mathcal{C}_1 \times \mathcal{C}_2$

In this section we will study the very ampleness of line bundles on the surface of the product of elliptic curves. It is well known that every k-very ample line bundle is $k-1$-very ample, then we will define a map

$$\phi : \mathrm{Amp}(S) \to \mathbb{Z}_{\geq -1}$$

which takes each ample line bundle L into the least integer k such that L is k-very ample but not $(k+1)$-very ample. We will denote $\phi(L)$ by $\phi(n_1, n_2)$ where n_1, n_2 are the powers of L_1, L_2 respectively.

4.2.1 Principal polarization Line Bundle

Definition 4.2.1. *We say a 0-dimensional X is **axial** if there is an $x \in S$ such that $X \subset \tau_x \mathcal{C}_1$ or $X \subset \tau_x \mathcal{C}_2$.*

Lemma 4.2.2. *Let $L = L_1^{n_1} \otimes L_2^{n_2}$ where $1 \leq n_2 \leq n_1$ and X axial, then $L_1^{n_1} \otimes L_2^{n_2} \otimes \mathcal{I}_X$ is not* IT_0 *for any X with length $|X| \geq n_2$.*

Proof. Without loss of generality let $X \subset \mathcal{C}_1$, then we have short exact sequence

$$0 \to L_1^{n_1-1} \otimes L_2^{n_2} \to L_1^{n_1} \otimes L_2^{n_2} \otimes \mathcal{I}_X \to Q \to 0 \tag{4.2.1}$$

where Q is the quotient sheaf with Chern character $\mathrm{ch}(Q) = (0,(1,0), n_2 - |X|)$. Since $L_1^{n_1-1} \otimes L_2^{n_2}$ is IT_0 for $n_1 \geq 2$ and $n_2 \geq 1$, then $\chi(L_1^{n_1-1} \otimes L_2^{n_2}) > 0$. Apply Φ to (4.2.1) we get $\Phi^1(L_1^{n_1} \otimes L_2^{n_2} \otimes \mathcal{I}_X) \cong \Phi^1(Q)$. Lemma 1.5.8 implies that Q is WIT_1 if $n_2 - |X| \leq 0$. Hence $L_1^{n_1} \otimes L_2^{n_2} \otimes \mathcal{I}_X$ is not WIT. □

Proposition 4.2.3. *Let $L = L_1^{n_1} \otimes L_2^{n_2}$ where $1 \leq n_2 \leq n_1$, then $\phi(n_1, n_2) \leq n_2 - 2$.*

The following theorem gives the value of $\phi(n_1, n_2)$ in the case $n_1 = n_2 = n \geq 2$.

Theorem 4.2.4. *If $L = L_1 \otimes L_2$, then $\phi(n,n) = n-2$ for $n \geq 2$*

Proof. We will induct on n to prove this theorem. Since L^2 is generated by global section then the statement is true if $n = 2$. Assume$X$ is a zero scheme with length $|X| = n-1$ and $X \setminus X'$ lies on D_L such that $X \setminus X' = \{x\}$. Then we have a short exact sequence

$$0 \to L_1^{n-1} \otimes L_2^{n-1} \otimes \mathcal{I}_{X'} \to L_1^{n} \otimes L_2^{n} \otimes \mathcal{I}_X \to Q \to 0 \tag{4.2.2}$$

where Q is the quotient sheaf with Chern character $\mathrm{ch}(Q) = (0,(1,1), 2n-2)$. Without loss of generality, we can assume that $X \cap \mathcal{C}_2 = \phi$. Applying $- \otimes \mathcal{O}_{\mathcal{C}_2}$ to (4.2.2), we obtain this diagram

$$\begin{tikzcd}
 & & & Q|_{\mathcal{C}_1} & \\
0 \arrow[r] & L_1^{n-1} \otimes L_2^{n-1} \otimes \mathcal{I}_{X'} \arrow[r] \arrow[dr] & L_1^{n} \otimes L_2^{n} \otimes \mathcal{I}_X \arrow[r] \arrow[ur] & Q \arrow[r] \arrow[u] & 0 \\
 & & K' \arrow[u] \arrow[r] & K \arrow[u] &
\end{tikzcd}$$

where K is supported on $\mathcal{C}_1$ and $K' \cong L_1^{n-1} \otimes L_2^{n} \otimes \mathcal{I}_{X''}$ for $X'' = X'$. Therefore K' has Chern character $\mathrm{ch}(K') = (1,(n-1,n),(n-1)^2)$, then $Q|_{\mathcal{C}_2}$ has Chern character

$\mathrm{ch}(Q|_{\mathcal{C}_1}) = (0,(1,0), n-1)$ and $\mathrm{ch}(K) = (0,(0,1), n-1)$ then Lemma 4.1.6 implies that $Q|_{\mathcal{C}_1}$ and K are IT_0 and hence Q is IT_0. □

The following theorem proves that the upper pound of $\phi(n_1, n_2)$ in Proposition 4.2.3 is sharp.

Theorem 4.2.5. *Let* $L = L_1^{n_1} \otimes L_2^{n_2}$*, then* $\phi(n_1, n_2) = \min(n_1, n_2) - 2$.

Proof. Without lose of generality, let $1 \le n_2 \le n_1$. By Proposition 4.2.3 we need to show that $\phi(n_1, n_2) \ge n_2 - 2$. To prove this theorem we will induct on n_1. Theorem 4.2.4 shows that the statement is true if $n_1 = n_2$. Assume that $L_1^{n_1-1} \otimes L_2^{n_2} \otimes \mathcal{I}_{X'}$ is IT_0 for all X' of length $|X'| = n_2 - 1$. Let $X \cap \mathcal{C}_1 = \phi$. Then we have short exact sequence

$$0 \to \mathcal{P}_{x_2} \otimes L_1^{n_1-1} \otimes L_2^{n_2} \otimes \mathcal{I}_{X'} \hookrightarrow L_1^{n_1} \otimes L_2^{n_2} \otimes \mathcal{I}_{X'} \to \mathcal{O}_{\mathcal{C}_1} \otimes L_1^{n_1} \otimes L_2^{n_2} \otimes \mathcal{I}_{X'} \to 0 \tag{4.2.3}$$

By the induction hypotheses $\mathcal{P}_{x_2} \otimes L_1^{n_1-1} \otimes L_2^{n_2} \otimes \mathcal{I}_{X'}$ is IT_0 and also $\mathcal{O}_{\mathcal{C}_1} \otimes L_1^{n_1} \otimes L_2^{n_2} \otimes \mathcal{I}_{X'} \cong \mathcal{O}_{\mathcal{C}_1} \otimes L_1^{n_1} \otimes L_2^{n_2}$ is IT_0 as well. Thus $L_1^{n_1} \otimes L_2^{n_2} \otimes \mathcal{I}_{X'}$ is IT_0 which means that $\phi(n_1, n_2) \ge n_2 - 2$. □

Proposition 4.2.6. *If* $L = L_1 \otimes L_2^2$ *(or* $L = L_1^2 \otimes L_2$*) and* $n_i = 1$ *, then* $\phi(1,2) = -1$ *(and* $\phi(2,1) = -1$*). In particular, L is not k-very ample for all integers* $k \ge 0$.

Proof. Let Q be a zero scheme with length $|Q| = 2$ and $x \in Q$, then we have a short exact sequence

$$0 \to L_1 \otimes L_2^2 \otimes \mathcal{I}_x \to L_1^2 \otimes L_2^2 \otimes \mathcal{I}_Q \to T \to 0 \tag{4.2.4}$$

where T is the quotient sheaf with Chern character $\mathrm{ch}(T) = (0,(1,0),1)$. Lemma 4.1.6 implies that T is IT_0, but $L_1^2 \otimes L_2^2 \otimes \mathcal{I}_Q$ is not WIT_0. Hence t $L_1 \otimes L_2^2 \otimes \mathcal{I}_x$ is not WIT_0 and so $\phi(1,2) = -1$. By symmetry it follows that $\phi(2,1) = -1$. □

4.2.2 $(1,d)$-Polarization Line Bundle

In this section we will generalize our result in Theorem 4.2.5 and study the ampleness of $(1,d)$-polarization line bundles. The question now is how can we use our result in the previous section to get the value of ϕ for $(1,d)$-polarization line bundles on an abelian surface isogenous to a product?.

Let $\tilde{S}$ be a $(1,d)$-polarization abelian surface and f an isogeny map from $\tilde{S}$ into S

$$f : \tilde{S} \to S$$

where $S = \mathcal{C}_1 \times \mathcal{C}_2$ a surface of the product of two elliptic curves $\mathcal{C}_1$ and $\mathcal{C}_2$. Consider the Néron-Severi group $\mathrm{NS}(S) =< L_1, L_2 >$ with $c_1^2(L_1) = 2\alpha$ and $c_1^2(L_2) = 2\beta$, then by taking the pullback of f we get $\mathrm{NS}(\tilde{S}) =< f^*L_1, f^*L_2 >$. Denote f^*L_i by $\tilde{L}_i$ for $i = 1,2$.

For any ample line bundle $\tilde{L}$ on $\tilde{S}$, $\tilde{L} = \tilde{L}_1^{n_1} \times \tilde{L}_2^{n_2}$ where $\tilde{L}_i$ is a line bundle corresponding to $\tilde{\mathcal{C}_i} = f^*\mathcal{C}_i$ for $i = 1,2$ and $\tilde{\mathcal{C}_1} \cdot \tilde{\mathcal{C}_2} = d$. To avoid any confusions we will denote $\phi(\tilde{L})$ by $\tilde{\phi}(n_1, n_2)$ where n_i is the power of $\tilde{L}_i$ for $i = 1,2$. The next lemma gives us the upper bound of $\tilde{\phi}(\tilde{L})$.

Lemma 4.2.7. *Let $\tilde{S}$ be a $(1,d)$-polarization abelian surface and $\tilde{L}$ an ample line bundle on it, then*

$$\tilde{\phi}(n_1, n_2) \le d(\min(n_1, n_2) - 1).$$

Proof. Suppose that $\tilde{L}$ is not $\tilde{k}$-very ample, then there is a 0-dimensional subscheme $\tilde{X}$ with length $|\tilde{X}| = \tilde{k} + 1$ such that $\tilde{L}^n \otimes \mathcal{I}_{\tilde{X}}$ is not WIT_0. Hence $H^1(\tilde{L}^n \otimes \mathcal{I}_{\tilde{X}}) \neq 0$ which implies that $H^1(L^n \otimes \mathcal{I}_X) \neq 0$ where $\tilde{X} = f^{-1}(X)$. If $|X| = k$, then $|\tilde{X}| = dk$ and $\tilde{\phi}(n_1, n_2) + 2 \le d\phi(n_1, n_2)$. Hence $\tilde{\phi}(n_1, n_2) \le d(\min(n_1, n_2) - 1)$. □

Theorem 4.2.8. *If $\tilde{L} = \tilde{L}_1 \otimes \tilde{L}_2$, then $\tilde{\phi}(n,n) = dn - 2$ for $n \ge 2$*

Proof. The proof of this theorem is similar to the proof of Theorem 4.2.4. The Chern character of Q in (4.2.2) is $\mathrm{ch}(Q) = (0, (1,1), 2\alpha(2n-1) + dn - |X|)$ where $|X| = n - 1$.

Now restrict Q in $\tilde{\mathcal{C}}_2$, to get the following diagram

$$\begin{tikzcd}
& & & Q|_{\tilde{\mathcal{C}}_1} & \\
0 \arrow[r] & \tilde{L}_1^{n-1} \otimes \tilde{L}_2^{n-1} \arrow[r] \arrow[dr] & \tilde{L}_1^{n} \otimes \tilde{L}_2^{n} \otimes \mathcal{I}_X \arrow[r] \arrow[ur] & Q \arrow[r] \arrow[u] & 0 \\
& & K' \arrow[u] \arrow[r] & K \arrow[u] &
\end{tikzcd}$$

where $\operatorname{ch}(Q|_{\tilde{\mathcal{C}}_1}) = (0,(0,1),dn-1)$ and $\operatorname{ch}(K) = (0,(0,1),d(n-1))$, and both are IT_0 which forces Q to be IT_0. $\square$

Lemma 4.2.9. *Let Z be a 0-dimensional subscheme of a polarized abelian surface $(\tilde{S},\ell)$ with $\ell^2 = 2d$ and $|Z| \leq d+1$, then Z is collinear.*

Proof. Consider the case $|Z| = d+1$. If $\tilde{L}$ is a polarization line bundle corresponds to ℓ, then $\chi(\tilde{L}\mathcal{I}_Z) = d - |Z| = -1$. If Z is not collinear then $\Phi^0(\tilde{L}\mathcal{P}_{\hat{S}}\mathcal{I}_Z) = 0$ for all $\hat{x} \in \hat{\hat{X}}$ and $H^0(\tilde{L}P_{\hat{x}}\mathcal{I}_Z) = 0$. Then $\tilde{L}\mathcal{I}_Z = 0$ is IT_1 with Chern character $\operatorname{ch}(\Phi^1(\tilde{L}\mathcal{I}_Z)) = (1,\ell,-1)$ and this contradicts that $\Phi^1(\tilde{L}\mathcal{I}_Z)$ is locally free. Hence Z is collinear. If $|Z| < d+1$, then $\chi(\tilde{L}\mathcal{I}_Z) > 0$ and $H^2(\tilde{L}\mathcal{I}_Z\mathcal{P}_{\hat{x}}) = 0$. So $H^0(\tilde{L}\mathcal{I}_Z\mathcal{P}_{\hat{x}}) \neq 0$ for all $\hat{x}$. $\square$

Theorem 4.2.10. *Let $\tilde{L} = \tilde{L}_1^{n_1} \otimes \tilde{L}_2^{n_2}$, then $\tilde{\phi}(n_1,n_2) = d\min(n_1,n_2) - 2$.*

Proof. We want to prove that the upper bound in Lemma 4.2.7 is sharp. Without lose of generality, let $1 < n_2 \leq n_1$ and we will induct on n_1.

The statement is true if $n_1 = n_2$ by Theorem 4.2.8. Suppose that $\tilde{L}_1^{n_1} \otimes \tilde{L}_2^{n_2} \otimes \mathcal{I}_X$ is IT_o for all 0-dimensional subscheme X of length $|X| < dn_2 - 1$. Let $X"$ and X' be 0-dimensional subschemes with $|X'| = dn_2 - 1$ and $|X"| = d(n_2 - 1)$ such that $X" \subset X'$ and $X' \setminus X"$ lies on $D_{\tilde{L}}$. This exists by Lemma 4.2.9 as $|X' \setminus X"| = d-1$. Then we have a short exact sequence

$$\tilde{L}_1^{n_1-1} \otimes \tilde{L}_2^{n_2} \otimes \mathcal{I}_{X"} \to \tilde{L}_1^{n_1} \otimes \tilde{L}_2^{n_2} \otimes \mathcal{I}_{X'} \to Q \tag{4.2.5}$$

where Q is the quotient sheaf with $\operatorname{ch}(Q) = (0,(1,0),d(n_2-1)+1)$. Since $\chi(Q) > d+1$

when $n_2 > 1$, then Q is IT_0. Moreover, from induction hypothesis $\tilde{L}_1^{n_1-1} \otimes \tilde{L}_2^{n_2} \otimes \mathcal{I}_{X"}$ is IT_0 and then $\tilde{L}_1^{n_1} \otimes \tilde{L}_2^{n_2} \otimes \mathcal{I}_{X"}$ is IT_o for all $X"$ of length $|X"| \leq dn_2 - 1$. Hence $\tilde{\phi}(n_1, n_2) = d\min(n_1, n_2) - 2$. $\square$

Chapter 5

Moduli Spaces of Bridgeland Stable Objects

Let (S,ℓ) be an abelian surface. Recall "the Large Volume Limit Theorem":

Proposition 5.0.11. *Fix a pair $(s,t)\in\mathbb{H}$. Suppose that $E\in\mathcal{A}_s$ satisfies $r(E)>0$ and $(c_1(E)-r(E)s).t>0$. Then E is σ_t-semistable as $t\to\infty$ (i.e. there is t_0 such that for each $t>t_0$ E is σ_t-semistable) if and only if E is a shift of (s,t)-twisted semistable sheaf on S.*

Proof. See [Bri08, Prop 14.2]. □

So as $t\to\infty$, the σ_t-semistable objects E with $\mu(E)>0$ are exactly the Gieseker semistable sheaves (when $s=0$). The case when $\mu(E)<0$ is similar. In this case, $r(E)<0$ when $s=0$:

Proposition 5.0.12. *Suppose $E\in\mathcal{A}_0$ with $\mu(E)<0$. If E is σ_t-semistable for all $t\gg 0$, then $H^0(E)$ is supported in dimension 0 and $H^{-1}(E)$ is Gieseker semistable vector bundle.*

Proof. This proof follows in the same way as the proof of [Bri08, Prop 14.2]. If E is Bridgeland stable for all $t\gg 0$ then $H^0(E)$ must be supported in dimension 0, otherwise $\mu_{0,t}(H^0(E))$ is finite and then $H^0(E)$ destabilizes E for $t \gg 0$. Since

$$0 \to \mathcal{O}_Z \to H^{-1}(E)[1] \to H^{-1}(E)^{**}[1] \to 0$$

is a short exact sequence in $\mathcal{A}_0$ and then $\mathcal{O}_Z \to E$ would destabilize E. The fact that $H^{-1}(E)$ is Gieseker semistable follows in the same way as [Bri08]. □

Remark 5.0.13. *For an alternative proof of the previous proposition, we can use an observation of Yanagida and Yoshioka who show that the* $[1] \circ \Delta$ *preserves Bridgeland stability, where* $\Delta(E) = \mathbf{R}\mathcal{H}om(E, \mathcal{O}_S)$ *(see [YY14, Prop 2.9]) at least when* $c_1 \cdot \ell \neq 0$. *So if* F *is* σ_t*-semistable then* $F^\vee$ *is* σ_t*-semistable and* $\mu(F^\vee) > 0$. *Then Prop. 14.2 in [Bri08] implies that* $F^\vee$ *is a Gieseker semistable sheaf. Therefore* $F^{\vee\vee} \cong F$ *takes the required form. In particular, observe that* $H^0(F) \neq 0$ *exactly when* $F^\vee$ *is not locally-free.*

Proposition 5.0.14. *([Huy08] and [MM13, Prop 3.2])*

1. $\Phi[1]$ *preserves* $\mathcal{A}_0$, *and*

2. $E \in \mathcal{A}_0$ *is* σ_t*-semistable if and only if* $\Phi[1](E)$ *is* $\sigma_{1/dt}$*-semistable.*

Proof. 1. (See[Huy08]).

2. The case $d = 1$ is in [MM13, Prop 3.2]), we can generalize the argument to other values of d.

$$\begin{aligned}\Phi[1]((1, -ti\ell, -t^2 d)) &= (t^2 d, -ti\ell, -1)\\ &= t^2 d(1, -\frac{1}{td} i\ell, \frac{1}{t^2 d})\\ &= t^2 d \exp(-\frac{1}{td}\ell i).\end{aligned}$$

Hence $\Phi[1]\exp(-t\ell i) = t^2 d \exp(-\frac{1}{td}\ell i)$.

□

5.1 Walls and Pseudo-Walls

By a pseudo-wall we mean a wall determined by the numerical condition but without (necessarily) requiring a destabilizing object to exist. On some surfaces, pseudo-walls can exist which are not actual walls.

In this section we want to prove that the pseudo-walls are real by using the trick that was used in [BM14][1]. In this paper Bayer and Macrì showed that $K_{\mathbb{Q}}(S) = \mathbb{Q}^3$ and $\chi(-,-)$ is of type $(1,2)$. Now suppose S is a projective surface with $K_S = 0$. Assume E with $\mathrm{ch}(E) = \boldsymbol{v}$ is $\sigma_{s,t}$-stable for all $t > t_0$ and is destabilized by an $\mathcal{A}_s$ short exact sequence at t_0

$$0 \to F \to E \to G \to 0.$$

Definition 5.1.1. *We let $\mathcal{M}_{\mathrm{ch}}^{\sigma_{s,t}}$ denote the moduli space of σ_t-semistable objects in $\mathcal{A}_0$.*

Recall the definition of the Jordan-Hólder filtration:

Definition 5.1.2. *A Jordan-Hólder filtration of a semi-stable object E is a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E,$$

such that the factors E_i/E_{i-1} are $\sigma_{s,t}$-stable for all $i = 1,..,k$ with reduced Hilbert polynomial $p(E)$.

It splits E into its stable components. The value of k is the length of the Jordan-Hólder filtration. We let $JH_1^{\sigma_{s,t}} \subset \mathcal{M}_{\mathrm{ch}}^{\sigma_{s,t}}$ denote the locus of equivalence classes of objects in $\mathcal{A}_{s,t}$ of length 1.

The following lemma shows that we can chose F and G to be $\sigma_{s,t}$-semistable at $t = t_0$.

Lemma 5.1.3. *The moduli space $\mathcal{M}_{\boldsymbol{v}}^{GS}$ is non empty if and only if $\boldsymbol{v}$ satisfies the Bogomolov inequality.*

[1]We are grateful to Arend Bayer for pointing this out.

Proof. See [Yos01]. □

Lemma 5.1.4. *For each* $(s,t) \in \mathbb{H}$ *and a Mukai vector* $\boldsymbol{v}$ *satisfying the Bogomolov inequality, we have* $\mathcal{M}_{\boldsymbol{v}}^{\sigma_{s,t}} \neq \phi$.

Proof. Since $K_S = 0$, $\mathcal{M}_{\boldsymbol{v}}^{\sigma_{s,t}} \neq \phi$ if and only if $\mathcal{M}_{\boldsymbol{v}}^{\sigma_{s,t'}} \neq \phi$ for any $t' > 0$. But for $t' >> 0$, $\mathcal{M}_{\boldsymbol{v}}^{\sigma_{s,t'}} = \mathcal{M}_{\boldsymbol{v}}^{GS}$ or $\mathcal{M}_{\boldsymbol{v}^*}^{GS}$. □

It is well known that the moduli space of σ_t-semistable objects in $\mathcal{A}_0$ are projective varieties for many selections of spaces. For the case of K3 and abelian surfaces it follows from the sliding down the wall trick of [Mac14] and applying Proposition 5.0.14.

Miramide, Yanagida and Yoshioka independently observed that if S is an abelian, or a $K3$ surface in the Picard rank 1 case, then $\mathcal{M}_{\mathrm{ch}}^{\sigma_{s,t}}$ is a smooth projective surface [MYY14, Theorem 3.3.3]. Furthermore, Bayer and Macri proved this more generally for K3 surfaces [BM14, Theorem 1.3].

On the other hand, the large volume limit (Proposition 5.0.11) for example shows that as $t \to \infty$, $\mathcal{M}_{\mathrm{ch}}^{\sigma_{s,t}} = \mathcal{M}_{\mathrm{ch}}^{\mathrm{GS}}$ when $c_1(\mathrm{ch}) \cdot \ell > 0$ and $r(\mathrm{ch}) > 0$. The notation of equality here means that the points in $\mathcal{M}_{\mathrm{ch}}^{\sigma_{s,t}}$ represent exactly the same objects in $\mathrm{Coh}(S) \cap \mathcal{A}_0$ up to isomorphism. When the degree is negative, Proposition 5.0.13 implies that the large volume limit of $\mathcal{M}_{\mathrm{ch}}^{\sigma_{s,t}}$ for large t is given by objects $E^\vee[1]$, where $[E] \in \mathcal{M}^{\mathrm{GS}}_{\mathrm{ch}^* := (ch_0, -ch_1, ch_2)}$ and so $\mathcal{M}_{\mathrm{ch}}^{\sigma_{s,t}} \cong \mathcal{M}_{\mathrm{ch}^*}^{\mathrm{GS}}$.

It may happen for some value of t that the two moduli spaces are not equal. In fact, there will be a strictly decreasing sequence $t_0, t_1, \ldots$ of values of t on either side of which $\mathcal{M}_{\mathrm{ch}}^{\sigma_{s,t}}$ differ. We call these walls (sometimes they are called mini-walls when we fix s). In particular, if we fix s, then for large $t \gg 0$ and Bogomolov vector $\boldsymbol{v}$, we have $\mathcal{M}_{\boldsymbol{v}}^{\sigma_{s,t}}$ birational to $\mathcal{M}_{\boldsymbol{v}}^{GS}$ or $\mathcal{M}_{\boldsymbol{v}^*}^{GS}$. As $\boldsymbol{v}$ crosses the wall W at t, we have two cases:

a. The first case is if codim W=0 :

This case means that all stable objects became unstable then there exists a Fourier-Mukai transform Φ such that $\mathcal{M}_{\boldsymbol{v}}^{\sigma_{s,t'}} = \Phi\mathcal{M}_{\boldsymbol{v}}^{\sigma_{s,t"}}$ where $t' < t < t"$ (See [MYY14]).

b. The second case is when codim W >0 :

Let $\mathcal{N}$ be the space of all stable objects E that became semistable after crossing the wall W. Then, E_1 destabilizes E with quotient Q_1. Let E, E_1 and Q_1have Chern characters $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_1$ and $\boldsymbol{v}_2$ respectively.

Suppose that E_1 and Q_1 are $\sigma_{s,t}$-stable , then for all $E_1 \in \mathcal{M}_{\boldsymbol{v}_1}$ and $Q_1 \in \mathcal{M}_{\boldsymbol{v}_2}$ we have $\operatorname{Hom}(E_1, Q_1) = \operatorname{Hom}(Q_1, E_1) = 0$. Therefor, $\dim \operatorname{Ext}^1(E_1, Q_1) = -\chi(E_1, Q_1) = \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle$ and $\dim \mathbb{P}\operatorname{Ext}^1(E_1, Q_1) = \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle - 1$. On the other hand,

$$\begin{aligned} \dim \mathcal{M}_{\boldsymbol{v}} &= 2 + \langle \boldsymbol{v}, \boldsymbol{v} \rangle \\ &= 2 + \langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle + \langle \boldsymbol{v}_2, \boldsymbol{v}_2 \rangle + 2\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle \end{aligned}$$

But $\dim \mathcal{M}_{\boldsymbol{v}_i} = 2 + \langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle$ for $i = 1, 2$ and $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \dim \mathbb{P}\operatorname{Ext}^1(E_1, Q_1) + 1$. Then

$$\dim \mathcal{M}_{\boldsymbol{v}} = \dim \mathcal{M}_{\boldsymbol{v}_1} + \dim \mathcal{M}_{\boldsymbol{v}_2} + 2 \dim \mathbb{P}\operatorname{Ext}^1(E_1, Q_1).$$

Hence $\operatorname{codim} \mathbb{P}\operatorname{Ext}^1(E_1, Q_1) = \dim JH_1^{\sigma_{s,t}}$. If $\operatorname{codim} JH_1^{\sigma_{s,t}} \geq 2$ so $\mathcal{N} \subset \mathcal{M}^{\sigma_{s,t"}}$, then $\mathcal{M}^{\sigma_{s,t"}}$ is birational isomorphic to $\mathcal{M}^{\sigma,t'}$ and if $\operatorname{codim} JH_1^{\sigma_{s,t}} = 1$, then $\mathbb{P}\operatorname{Ext}^1(E_1, Q_1) = \mathbb{P}^1$ and $\dim \mathcal{M}_{\boldsymbol{v}_2} = 0$ which is impossible.

Lemma 5.1.5. *If $E_1 \subset E$ is JH length 1 at t, then any extension $[\varepsilon] \in \operatorname{Ext}^1(Q_1, E_1)$*

$$0 \to E_1 \to E \to Q_1 \to 0$$

is $\sigma_{s,t''}$-stable for $t'' - t > 0$ sufficiently small and dually

$$0 \to Q_1 \to F \to E_1 \to 0$$

is $\sigma_{s,t'}$-stable for $t-t'>0$ sufficiently small.

Proof. The proof of this lemma follows from local finiteness of the walls. □

In other words, when $\boldsymbol{v}$ crosses the wall the moduli spaces may be different but the stratifications are the same, i.e. $JH_1^{\sigma_{s,t'}} \cong JH_1^{\sigma_{s,t''}}$. It is clear that this works inductively for all lengths of filtration and for each t there is finite disjoint union of Jordan-Hólder strata of the moduli space each of which survives going through the wall.

Lemma 5.1.6. *For $t-t'>0$ sufficiently small, $JH_k^{\sigma_{s,t'}} \cong JH_k^{\sigma_{s,t''}}$ as varieties (but not canonically).*

Corollary 5.1.7. *The number of connected components of $\mathcal{M}^{\sigma_{s,t}}$ does not change as we cross a wall.*

In other words, components of the Bridgeland stable moduli spaces cannot be "created" or "destroyed" as we cross a wall. In particular, to show that there are no walls for a given s and Chern character ch it suffices to start with a Gieseker stable sheaf (or its dual) and show that it is $\sigma_{s,t}$-stable for all t. We can go further when there are walls. At each wall there is a submoduli space M of $\sigma_{s,t}$-stable objects for $t \in (t_0, t_0+\epsilon)$ which become semistable on the wall $t=t_0$ and there is a dual space M^* of objects constructed as above.

Let $\boldsymbol{e} = \boldsymbol{e}^{(-s-it_0)\ell}$ then $Z(\boldsymbol{v}) = -<\boldsymbol{e}, \boldsymbol{v}>$ for all $\boldsymbol{v} = \mathrm{ch}(E)$, $E \in \mathcal{A}_s$ and $\Im(Z) \geq 0$.

Lemma 5.1.8. *If $E \in \mathcal{M}_{\boldsymbol{v}}^{\sigma_{s,t}}$ and there is a pseudo-wall given by independent $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ satisfy Bogomolov inequality, then $\chi(\boldsymbol{v}_1, \boldsymbol{v}_2) < 0$.*

The proof of this lemma is in the following technical lemmas

Lemma 5.1.9. *The Euler characteristic $\chi(\boldsymbol{v}_1, \boldsymbol{v}_2)$ is not zero.*

Proof. Suppose that $\chi(\boldsymbol{v}_1, \boldsymbol{v}_2) = 0$, then consider the plane spanned by $\boldsymbol{v}_1, \boldsymbol{v}_2$. Take

$\boldsymbol{w} = \lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2$ where $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$. Then

$$\begin{aligned}\chi(\boldsymbol{w}, \boldsymbol{v}_2) &= \chi(\lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2, \boldsymbol{v}_2)\\ &= \lambda_1 \chi(\boldsymbol{v}_1, \boldsymbol{v}_2) + \lambda_2 \chi(\boldsymbol{v}_2, \boldsymbol{v}_2)\\ &= \lambda_2 \chi(\boldsymbol{v}_2, \boldsymbol{v}_2) \leq 0, \quad \text{by Bogomolov.}\end{aligned}$$

Also, $\chi(\boldsymbol{v}_1, \boldsymbol{v}_1) \leq 0$. Hence $\chi(-,-) |_{<\boldsymbol{v}_1, \boldsymbol{v}_2>}$ is of type $(0,2)$, i.e. $\chi(\Im e, \boldsymbol{v}_i) \leq 0$ for $i \leq 1,2$. But if we take $\Im Z_{s,t_0}(\boldsymbol{w})$ we get $-\chi(\Im(\boldsymbol{e}), \boldsymbol{w}) > 0$, then $K(S) =< \boldsymbol{v}_1, \boldsymbol{v}_2, \Im(\boldsymbol{e}) >$ and that means $\chi(-,-)$ is of type $(0,3)$ and it is a contradiction. □

Lemma 5.1.10. *Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be the same as Lemma 5.1.8. If $\chi(\boldsymbol{v}_1, \boldsymbol{v}_1) = 0$, then $\chi(\boldsymbol{v}_2, \boldsymbol{v}_2) \neq 0$.*

Proof. Assume that $\chi(\boldsymbol{v}_2, \boldsymbol{v}_2) = 0$ and for λ_1 and $\lambda_2 \in \mathbb{R}_{>0}$ we have

$$\chi(\lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2, \boldsymbol{v}_1) = \lambda_1 \chi(\boldsymbol{v}_1, \boldsymbol{v}_1) + \lambda_2 \chi(\boldsymbol{v}_2, \boldsymbol{v}_1) = \lambda_2 \chi(\boldsymbol{v}_2, \boldsymbol{v}_1) \tag{5.1.1}$$

and,

$$\chi(\chi(\lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2, \boldsymbol{v}_2) = \lambda_1 \chi(\boldsymbol{v}_1, \boldsymbol{v}_2) + \lambda_2 \chi(\boldsymbol{v}_2, \boldsymbol{v}_2) = \lambda_1 \chi(\boldsymbol{v}_1, \boldsymbol{v}_2). \tag{5.1.2}$$

Hence $\chi(-,-)$ is semidefinte and that is contradiction. Then $\chi(\boldsymbol{v}_2, \boldsymbol{v}_2) \neq 0$. □

Going back to the proof of Lemma 5.1.8, suppose without loss of generality that $\chi(\boldsymbol{v}_2, \boldsymbol{v}_2) \leq 0$ and assume that $\chi(\boldsymbol{v}_1, \boldsymbol{v}_2) > 0$. Pick $0 \leq \lambda \leq 1$, then $\chi(\lambda \boldsymbol{v}_1 + (1-\lambda)\boldsymbol{v}_2, \boldsymbol{v}_2)$ is negative when $\lambda = 0$, positive when $\lambda = 1$ and zero for some $0 < \lambda < 1$ which contradicts our assumption. Hence $\chi(\boldsymbol{v}_1, \boldsymbol{v}_2) < 0$.

We now show that all pseudo-walls are real.

Theorem 5.1.11. *If $\boldsymbol{v}_1$ is a Bogomolov vector and destabilizes $\boldsymbol{v}$ at $(s,t) \in \mathbb{H}$, and $\boldsymbol{v}_2 = \boldsymbol{v} - \boldsymbol{v}_1$ then there exist F with $\mathrm{ch}(F) = \boldsymbol{v}_1$ and $F \hookrightarrow E$ in $\mathcal{A}_s$ where $\mathrm{ch}(E) = \boldsymbol{v}$.*

Proof. By Lemma 5.1.8, $\chi(\boldsymbol{v}_1, \boldsymbol{v}_2) < 0$ and also Lemma 5.1.3 shows that there exist F with $\mathrm{ch}(F) = \boldsymbol{v}_1$ and G with $\mathrm{ch}(G) = \boldsymbol{v}_2$, and for $0 < t''$there are $F, G \in \mathcal{A}_s$ by Lemma 5.1.4 such that F, G are $\sigma_{s,t''}$-stable for $t" - t$ sufficiently small. Since $\chi(\boldsymbol{v}_1, \boldsymbol{v}_2) < 0$ and $\dim \mathrm{Ext}^1(G, F) \neq 0$, then there exists a non-trivial extension $0 \to F \to E \to G \to 0$ and E is $\sigma_{s,t''}$-stable which realizes the wall. □

5.2 Computing Walls

Our aim in this section is to identify these walls when $\mathrm{ch} = (0, \ell, \chi)$ and $\mathrm{ch} = (1, n\ell, n^2 d - |X|)$. In [Mac14], Maciocia studies the existence of walls for Bridgeland stability conditions for general projective surfaces and in the same paper he proves the following proposition

Proposition 5.2.1. *If* (S, ℓ) *is a surface with* $\rho(X) = 1$ *and* $g' = 1$*, then there are no walls in the basic Bridgeland stability plane for* $\mathrm{ch} = (0, \ell, k\ell^2)$ *for all* $k \in \mathbb{Z}$.

Proof. See [Mac14, Prop 4.1]. □

The following lemma is for the first case when $\mathrm{ch} = (0, \ell, \chi)$ which generalizes [Mac14, Prop 4.1]:

Lemma 5.2.2. *There are no walls for the Chern character* $\mathrm{ch} = (r, \ell, \chi)$ *for any* $\chi, r \in \mathbb{Z}$ *in* $\mathcal{A}_0$.

Proof. For all $E \in \mathcal{A}_0$ we have $c_1(E) \cdot \ell \geq 0$. If $c_1(E) = 0$ then $\mu_{0,t}(E) = \infty$. Now suppose E has $c_1(E) = \ell$ and sits in a short exact sequence

$$0 \to K \to E \to Q \to 0 \tag{5.2.1}$$

in $\mathcal{A}_0$. Then we have two cases. The first case is when $c_1(K) = 0$, then $\mu_{0,t}(K) = \infty$ and so the sequence (5.2.1) destabilizes E for all t. The second case is when $c_1(Q) = 0$ and in this case the sequence (5.2.1) does not destabilize for any t. Hence there are no walls in $\mathcal{A}_0$. □

Corollary 5.2.3. *There are no walls for the Chern character* $\mathrm{ch} = (0, \ell, \chi)$ *for any* $\chi \in \mathbb{Z}$ *in* $\mathcal{A}_0$.

Remark 5.2.4. *Since there are no walls for* $\mathrm{ch} = (0, \ell, \chi)$*, then all Gieseker stable objects with Chern character* $\mathrm{ch} = (0, \ell, \chi)$ *are Bridgeland stable ones then we can say that*

$$\mathcal{M}^{\mathrm{GS}}_{(0,\ell,\chi)} = \mathcal{M}^{\sigma_{s,t}}_{(0,\ell,\chi)}$$

for all t *and* $s = 0$*. On the other hand, Proposition 5.0.14 shows that* $\Phi[1]$ *preserves Bridgeland stability in* $\mathcal{A}_0$*, then* $\mathrm{ch} = (\chi, -\ell, 0)$ *also has no walls for* $s = 0$*. Hence, for all* $t > 0$*,*

$$\mathcal{M}^{\sigma_{s,t}}_{(\chi,\ell,0)} = \mathcal{M}^{GS}_{(\chi,\ell,0)}, \quad \text{when } \chi \geq 0$$

and

$$\mathcal{M}^{\sigma_{s,t}}_{(\chi,\ell,0)} = \Delta\mathcal{M}^{GS}_{(-\chi,\ell,0)}[1], \quad \text{when } \chi < 0.$$

Definition 5.2.5. *We say that the moduli space* $\mathcal{M}^{BS,t}_{(r,c\ell,\chi)}$ *of Bridgeland stable sheaves of Chern character* $(r, c\ell, \chi)$ *satisfies* IT_0 *(respectively* WIT_0*) if and only if for each* E *representing an object of* $\mathcal{M}^{BS}_{(r,c\ell,\chi)}$*,* E *satisfies* IT_0 *(respectively* WIT_0*).*

Note that if $\mathcal{M}$ is a fine moduli space and $[E] \in \mathcal{M}$ then E is IT_0 if and only if all $F \in [E] \in \mathcal{M}$ are IT_0. This may not be true when the moduli space is not fine (and there exist non-Gieseker stable sheaves) because the IT_0 condition is not preserved by S-equivalence. However, the moduli spaces we consider below will all be fine.

The following technical result will be useful in the next section:

Lemma 5.2.6. *The moduli space* $\mathcal{M}^{\mathrm{GS}}_{(0,\ell,\chi)}$ *is* IT_0 *if and only if* $\chi \geq d + 1$.

Proof. We use Proposition 5.0.12, Remarks 5.0.13 and 5.2.4, and Lemma 5.2.2 to give isomorphisms

$$\mathcal{M}^{\mathrm{GS}}_{(0,\ell,\chi)} \overset{\Phi[1]}{\cong} \mathcal{M}^{\sigma_{s,t}}_{-(\chi,-\ell,0)} \overset{[1]\Delta}{\cong} \mathcal{M}^{\sigma_{s,t}}_{(\chi,\ell,0)} = \mathcal{M}^{\mathrm{GS}}_{(\chi,\ell,0)}$$

for all $t > 0$. Then $[E] \in \mathcal{M}^{\mathrm{GS}}_{(0,\ell,\chi)}$ is IT_0 if and only if $[\Phi(E)[1]] \in \mathcal{M}^{\sigma_{s,t}}_{-(\chi,-\ell,0)} \cap \mathcal{M}^{\mathrm{GS}}_{(\chi,-\ell,0)}[1]$ which holds if and only if $\Delta\Phi(E) \in \mathcal{M}^{\mathrm{GS}}_{(\chi,\ell,0)}$ is locally-free. But, since all representative sheaves of $\mathcal{M}^{\mathrm{GS}}_{(\chi,\ell,0)}$ must be μ-stable, we see that there are non-locally-free sheaves in $\mathcal{M}^{\mathrm{GS}}_{(\chi,\ell,0)}$ if and only if $\mathcal{M}^{\mathrm{GS}}_{(\chi,\ell,1)} \neq \emptyset$. This happens exactly when the Bogomolov inequality holds for the Chern character $(\chi,\ell,1)$, in other words when $\chi \leq d$ as required. □

5.3 Application to k-Very Ample

Let (S,ℓ) be a polarized abelian surface such that $\mathrm{NS}(S) = \langle \ell \rangle$ and $c_1(L)^2 = \ell^2 = 2d$. In this section we generalize the value of $\phi(n)$ that we found in Chapter 3. Combining Proposition 3.1.3 and Definition 5.2.5:

Proposition 5.3.1. *$\mathcal{M}^{BS,t}_{(1,n\ell,n^2d-k)}$ is IT_0 for all $t >> 0$ if and only if L^n is $(k-1)$-very ample. In particular, $\phi_L(n) \geq k-1$.*

Proposition 5.3.2. *Let (S,ℓ) be a polarized abelian surface such that* $\mathrm{NS}(S) = \langle \ell \rangle$ *and $c_1(L)^2 = 2d$, then $\phi_L(n) \leq 2(n-1)d - 2$ for $n > 1$.*

Proof. By Lemma 5.2.6, there is a Q with Chern character $\mathrm{ch}(Q) = (0,\ell,d)$ which is not IT_0. Since $\chi(L^{-n+1} \otimes Q) = d(3-2n) < 0$ for $n > 1$ we have $\mathrm{Ext}^1(Q, L^{n-1}) \neq 0$. Pick a non trivial extension

$$0 \to L^{n-1} \to E \to Q \to 0$$

and suppose $T \hookrightarrow E$ is its torsion subsheaf. Then we have the following diagram:

$$\begin{tikzcd}
 & L^{n-1} \arrow[r] \arrow[d, equal] & F \arrow[r] & Q/T & \\
0 \arrow[r] & L^{n-1} \arrow[r] & E \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u] & 0 \\
 & & T \arrow[r, equal] \arrow[u] & T \arrow[u] &
\end{tikzcd}$$

Then Q/T must be supported in dimension zero. But then $\mathrm{Ext}^1(Q/T, L^{n-1}) = 0$ and so $L^{n-1} \to F \to Q/T$ must split, which is impossible as F is torsion-free and Q/T is torsion. Hence $T = 0$. Then $E \cong L^n \otimes \mathcal{I}_X$ for some X of length $|X| = 2d(n-1)$ and E is not IT_0. $\square$

The following theorem proves that the upper pound of $\phi_L(n)$ in Proposition 5.3.2 is sharp.

Theorem 5.3.3. *Let (S, L) be a polarized abelian surface with $\mathrm{NS}(S) = \langle \ell \rangle$ and $c_1(L)^2 = 2d$, then $\phi(L^n) = 2d(n-1) - 2$.*

Proof. Proposition 5.3.2 provided the upper bound of $\phi_L(n)$. Then we need to show that $\phi_L(n) \geq 2(n-1)d - 2$, and we do this by showing that $\mathcal{M}^{\sigma_{s,t}}_{(1,n\ell,n^2d-k)}$ is IT_0 for all t and $k = 2d(n-1) - 1$. Suppose that $E \cong L^n \otimes \mathcal{I}_X$ where $|X| = 2d(n-1) - 1$ is not IT_0 and $\Phi(E)$ is σ_t-stable for all $t \gg 0$. Then $\hat{E}$ is a two-step complex such that $H^{-1}(\hat{E})$ is Gieseker stable and $H^0(\hat{E})$ is in the form $\mathcal{O}_Z$, by Proposition 5.0.12. The Chern character $\mathrm{ch}(H^{-1}(\hat{E})) = ((n-1)^2 d + d + 1, -n\ell, 1 + |Z|)$. By Bogomolov

$$\big((n-1)^2 d + d + 1\big)(1 + |Z|) \leq n^2 d. \tag{5.3.1}$$

Therefore $|Z| \leq \dfrac{2d(n-1) - 1}{dn^2 - 2d(n-1) + 1}$. But $d(n-2)^2 + 2 > 0$ and so

$$dn^2 - 2d(n-1) + 1 > 2d(n-1) - 1.$$

Hence, $|Z| < 1$. Therefore $H^0(\hat{E}) = 0$ and so E is IT_0. If E is σ_t-stable for all t, then it follows that $\Phi(E)$ is σ_t-stable for all t (and so also for $t \gg 0$). This happens when there are no walls. Unfortunately, there are walls in general. To finish the proof we will identify all the walls and show that all σ_t-semistable objects are IT_0 directly.

We now induct on $n \geq 2$. If $n = 2$, then $d(n-2) = 0$ and so there are no walls which proves the result for $n = 2$. Suppose that the statement is true for $n - 1 \geq 2$. i.e. $L^{n-1} \otimes \mathcal{I}_X$ is IT_0 for all X with $|X| = 2d(n-2) - 1$.

To prove that $L^n \otimes \mathcal{I}_X$ is IT_0 for all X with $|X| = 2d(n-1) - 1$, the only possible walls are given by $L^{n-1} \otimes \mathcal{I}_Y$ where $|Y| < 2d\big((n-1) - 1\big) - 1$ (see §5.4). Then there is a short exact sequence

$$0 \to L^{n-1} \otimes \mathcal{I}_Y \to L^n \otimes \mathcal{I}_X \to Q \to 0$$

By induction, $L^{n-1} \otimes \mathcal{I}_Y$ is IT_0 and the Chern characters read

$$0 \to (1,(n-1)\ell,(n-1)^2 d - |Y|) \to (1, n\ell, n^2 d - |X|) \to (0, \ell, n^2 d - |X| - (n-1)^2 d + |Y|) \to 0$$

Then $\chi(Q) = (2n-1)d + |Y| - 2d(n-1) + 1 = 1 + d + |Y| \geq d + 1$ and Lemma 5.2.6, implies that Q is IT_0 as well. Hence $L^n \otimes \mathcal{I}_X$ is IT_0 for all X with $|X| = 2d(n-1) - 1$. □

Going back to to [Ter98b], Terakawa studied this property by using divisors theory. In particular, he used Rieder's Theorem to prove the following

Theorem 5.3.4. *[Ter98b, Theorem 1.1] Let S be an abelian surface, L an ample line bundle on S and $k \in \mathbb{Z}_{\geq 0}$ then the following are equivalent:*

- *L is k-very ample;*
- *$c_1^2(L) \geq 4k+6$ and there is no effective divisor D satisfying the following*

$$2\sqrt{(2k+3)(p_a(D)-1)} \leq L \cdot D \leq 2p_a(D) + k - 1 \leq 2k+1, \tag{5.3.2}$$

  where the integer $p_a(D)$ is the arithmetic genus of D and it is defined as $p_a(D) = 1 - \chi(\mathcal{O}_D)$.

In particular, we can deduce Theorem 5.3.3 from [Ter98b, Theorem 1.1]. Suppose $\mathrm{NS}(S) = \mathbb{Z}\ell$ and $D = p\ell$. Assume $n \geq 2$ and for L^n, the condition 5.3.2 becomes

$$2\sqrt{(2k+3)(p^2\ell^2)} \leq np\ell^2 \leq p^2\ell^2 + k + 1 \leq 2k+1 \tag{5.3.3}$$

which is equivalent to

$$4k+6 \leq n^2\ell^2, p(n-p)\ell^2 \leq k+1 \text{ and } p \leq \sqrt{k/\ell^2} \tag{5.3.4}$$

We have

$$(n/2)^2 - (k/\ell^2)^2 = (n^2\ell^2 - 4k)/4\ell^2, p \le n/2 \text{ and } p(n-p)\ell^2 \le k+1. \tag{5.3.5}$$

Hence there is a p satisfying (5.3.2) if and only if $(n-1)\ell^2 \le k+1$ and $\ell^2 \le k$. Now assume that $(n-1)\ell^2 \ge k+2$. Thus $\ell^2 \ge (k+2)/(n-1)$ implies that

$$\begin{aligned} n^2\ell^2 - (4k+6) &\ge \frac{n^2(k+2)}{n-1} - (4k+6) \\ &> \frac{n^2(k+2) - (n-1)(4k+8)}{n-1} \\ &= \frac{(n-2)^2(k+2)}{n-1} \ge 0 \end{aligned}$$

Therefore, $(n-1)\ell^2 \ge k+2$ implies k-very ample of L^n. But $\ell^2 = 2d$, then L^n is k-very ample if $k \le 2d(n-1)-2$.

Assume that $(n-1)\ell^2 \le k+1$, then $\ell^2 \le k$ or $n=2$ and $\ell^2 = k+1$ which implies that $c_1(L^n)^2 = 4n\ell^2 = 4n(k+1) < n(4k+6)$. Hence L^n is not k-very ample and [Ter98b, Theorem 1.1] implies Theorem 5.3.3[2].

5.4 Walls for $L^n \otimes \mathcal{I}_X$

To identify all walls for $E = L^n \otimes \mathcal{I}_X$ as $s=0$, suppose F with $\mathrm{ch}(F) = (r, c\ell, \chi)$ destabilizes E. Therefore, we have a short exact sequence in $\mathcal{A}_0$

$$0 \to F \to E \to Q \to 0$$

where Q is the quotient sheaf. In cohomology we get a long exact sequence in $\mathrm{Coh}(S)$

$$0 \to H^{-1}(Q) \to F \to E \to H^0(Q) \to 0$$

[2] We are grateful to the referee of [AM16] who pointed this out .

The slope of E where $\mathrm{ch}(E) = (1, n\ell, n^2d - |X|)$ is given by

$$\mu_{s,t}(E) = \frac{n^2d - |X| - 2dns - d(t^2 - s^2)}{2td(n-s)}.$$

Note that $n > s$ as $E \in T_s$. We also have

$$\mu_{s,t}(F) - \mu_{s,t}(E) > 0.$$

From (1.5.10) the destabilizing condition is given by

$$n\chi - c(n^2d - |X| - dt^2) - dnrt^2 > 0. \tag{5.4.1}$$

Therefore

$$n\chi - cn^2d + c|X| > (nr - c)dt^2,$$

and $c \leq nr$ because $\mu(H^{-1}(Q)) \leq 0$ and $\mu(F) \leq \mu(F/H^{-1}(Q)) \leq \mu(E)$. Hence a necessary condition for the existence of such a destabilizing object is

$$n\chi - cn^2d + c|X| > 0. \tag{5.4.2}$$

We will use this condition to identify all possible destabilizers of $L^n \otimes \mathcal{I}_X \in T_s$.

Lemma 5.4.1. *For $s = 0$, we have*

- *If $F \in \mathcal{A}_0$ destabilizes a sheaf E with $\mathrm{ch}(E) = (1, n\ell, \chi)$ where $\chi = n^2d - 2d(n-1) - 1$ and $n > 1$, then F is a sheaf and $r(F) \geq 1$.*

- *If $E = L^n \otimes \mathcal{I}_X$ with $|X| = 2d(n-1) - 1$, then F is a rank 1 torsion-free sheaf.*

Proof. To prove the first statement, let $F \in \mathcal{A}_0$ destabilize E, then there is a short exact sequence

$$0 \to F \to E \to Q \to 0$$

where $Q = E/F$ in $\mathcal{A}_0$.Take cohomology then we see that $H^{-1}(F) = 0$ and $F := H^0(F) \cong F$. Thus we have a long exact sequence in $\mathrm{Coh}(S)$:

$$0 \to H^{-1}(Q) \to F \to E \to H^0(Q) \to 0 \tag{5.4.3}$$

Suppose $\mathrm{ch}(F) = (r, g'\ell, \chi)$. Since $H^{-1}(Q) \in \mathcal{F}_0$ and $F \in \mathcal{A}_0$, then $\mu(E) \geq 0 \geq \mu(H^{-1}(Q))$. Then there is an integer $g > 0$ such that $c_1(F) = (nr - g)\ell$. Then $c_1(H^{-1}(Q)) = (nr - g - n + m)\ell \leq 0$ where $m \geq 0$. Therefore $0 < nr - g \leq n - m \leq n$. Hence, $c_1(F)$ can be written as $c_1(F) = (n - c)\ell$ for some positive integer $c < n$.

Now suppose $r(F) \neq 0$. Since we can assume E is Bridgeland stable it must be simple and so the Bogomolov inequality holds (as this is just the statement that the moduli space of simple torsion-free sheaves has dimension at least 2) and so we can write

$$\chi(F) = \frac{(n-c)^2 d}{r} - k,$$

for some rational number $k \geq 0$. Since F is a destabilizer of E, then from (1.5.10), we get

$$\big((n-c)^2 d - kr\big)n - \chi(n-c)r > dt^2(n + c(r-1)) > 0, \tag{5.4.4}$$

Rearranging it we obtain

$$(n-c)\big((n-c)d - \chi r\big) > krn > 0 \tag{5.4.5}$$

As $n - c > 0$, then we get walls if $-(n-1)^2 dr - dr - r + dn^2 - cdn > 0$ so dividing by $-n^2 dr$ we obtain

$$\frac{1}{r} > \left(1 - \frac{1}{n}\right)^2 + \frac{1}{n^2} + \frac{1}{dn^2} \geq \frac{1}{2}$$

for all n. Since $d > 0$, hence $r = 1$. In the same way we can prove the second statement and taking cohomology of the destabilizing sequence shows that $H^{-1}(F) = 0$, $F := H^0(F) \cong F$ and F must be torsion-free. $\square$

we have just proved that the Chern character of any destabilizer F of $L^n \otimes \mathcal{I}_X$ is given by $\mathrm{ch}(F) = \big(1, (n-c)\ell, (n-c)^2 d - k\big)$ which means that F is in the form $L^{n-c} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y$, for some $\hat{x} \in \hat{S}$ and Y has length $|Y| = k$. In the next following lemmas we will find c and k.

Lemma 5.4.2. *If $L^{n-m} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y$ destabilizes $L^n \otimes \mathcal{I}_X$ where $|X| = 2d(n-1)-1$, then $m = 1$.*

Proof. We assume, without loss of generality, that $\hat{x} = 0$. Suppose that $F = L^{n-m} \otimes \mathcal{I}_Y$ with $|Y| = k$ destabilizes $E = L^n \otimes \mathcal{I}_X$, then $\mu(F) - \mu(E) \geq 0$. But if $\mu(E) = \mu(F)$ then $\mu_{0,t}(E/F) = \infty$ and so F does not destabilize. Therefore from condition (5.4.5), we get

$$(n-m)\big(-(n-1)^2 dr - dr - r + dn^2 - dnm\big) > krn.$$

We get walls if and only if

$$\left(1 - \frac{m}{n}\right)(2dn - dnm - 2d - 1) > k \geq 0. \tag{5.4.6}$$

Since $1 - \dfrac{m}{n}$ is positive, this happens if and only if $2dn - dnm - 2d > 1$ and then $2 \geq 2 - \dfrac{d+1}{nd} > m > 0$. Hence, $m = 1$. □

Lemma 5.4.3. *If $L^{n-1} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y$ destabilizes $L^n \otimes \mathcal{I}_X$ for some X where $|X| = 2d(n-1)-1$, then $|Y| < d(n-2)-1 \leq 2d(n-2)-1$.*

Proof. Without loss of generality we assume $\hat{x} = 0$. Take F, E as in Lemma 5.4.2, then from (5.4.6) we get:

$$\left(1 - \frac{1}{n}\right)(dn - 2d - 1) > |Y| \geq 0 \tag{5.4.7}$$

Since $0 < 1 - \dfrac{1}{n} < 1$, then $dn - 2d - 1 > |Y|$. □

The following theorem proves that the destabilizer in Lemma 5.4.2 is the only destabilizer for $E \cong L^n \otimes \mathcal{I}_X$.

Theorem 5.4.4. *When $s = 0$, all walls for* $\mathrm{ch} = (1, n\ell, n^2d - 2d(n-1)+1)$ *are given by sheaves*

$$L^{n-1} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y \text{ with } |Y| \leq \left\lfloor \frac{(n-1)(d(n-2)-1)}{n} \right\rfloor,$$

and all σ_t-stable objects are of the form:

1. $L^n \mathcal{I}_X$ *where* $|X| = 2d(n-1)-1$, *or*
2. $0 \to Q \to E \to L^{n-1} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y \to 0$, *where* $Q \cong L^n\mathcal{I}_X / L^{n-1} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y$.

Proof. Consider $\tilde{E}$ with $\mathrm{ch}(\tilde{E}) = (1, n\ell, n^2d - 2d(n-1)+1)$. We know that $\tilde{E}$ is σ_t-stable for $t \gg 0$ if and only if $E \cong L^n \otimes \mathcal{I}_X$ where $|X| = 2d(n-1)-1$. Suppose there is a wall for $\tilde{E}$ at $t = t_0$, then from the previous lemmas, the strata $\mathcal{M}_{(1,n\ell,n^2d-2d(n-1)+1)}$ is replaced with objects of the form

$$0 \to Q \to E \to L^{n-1} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y \to 0 \tag{5.4.8}$$

where $Q \cong L^n\mathcal{I}_X / L^{n-1} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y$.

Suppose F is σ_t-stable with $\mathrm{ch}(F) = (r, (n-m)\ell, \chi)$ and destabilizes E in $\mathcal{A}_0$. Then there is a short exact sequence $0 \to F \to E \to \tilde{Q} \to 0$ and $\tilde{Q}$ is σ_{t_1}-semistable. If we take the cohomology we obtain a long exact sequence

$$0 \to H^{-1}(\tilde{Q}) \to F \xrightarrow{f} E \to H^0(\tilde{Q}) \to 0 \tag{5.4.9}$$

where $\tilde{Q}$ is the quotient and it is pure and $H^0(F) \leq 0$ because E is a sheaf. Furthermore, Lemma 5.4.1 show that $r(F) = 1$. Suppose the map f factors through F' and T is a torsion inside F', then we get the following diagram

$$\begin{tikzcd}
 & L^{n-m} \otimes \mathcal{I}_{Y'} \arrow[r] & L^{n-1} \otimes \mathcal{I}_Y \arrow[r] & B \\
F \arrow[r] & F' \arrow[r] \arrow[u] & E \arrow[r] \arrow[u] & H^0(\tilde{Q}) \arrow[u] \\
 & T \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u] & Q/T \arrow[u]
\end{tikzcd}$$

where T is a subsheaf of Q and $T \neq 0$, then T is pure and $c_1(T) = \ell$. Hence $Q/T = \mathcal{O}_Z$. If $r(F') = 1$, then $r(H^{-1}(\tilde{Q})) = 0$ but $H^{-1}(\tilde{Q}) \in \mathcal{F}_0$. Hence $H^{-1}(\tilde{Q}) = 0$ so $\tilde{Q} = H^0(\tilde{Q})$, $\mu_t(\mathcal{O}_Z) = \infty$ and $\tilde{Q}$ is assumed Bridgeland semistable. Hence $T = Q$.

Assume that $\tilde{E}$ is σ_{t_1}-stable where $t_1 > t_0$, then we have a short exact sequence

$$0 \to L^{n-1} \otimes \mathcal{I}_Y \to \tilde{E} \to Q \to 0. \tag{5.4.10}$$

From the digram we have

$$\mu_{t_0}(L^{n-m} \otimes \mathcal{I}_{Y'}) \geq \mu_{t_0}(F) \geq \mu_{t_0}(E)$$

but $\mu_{t_0}(E) = \mu_{t_0}(\tilde{E})$.Therefore $L^{n-m} \otimes \mathcal{I}_{Y'}$ destabilizes $\tilde{E}$ but Lemma 5.4.2 shows that $m = 1$, then$\tilde{Q} = \mathcal{O}_Z$ as it does not destabilize E.

If $r(F') = 0$, then $H^{-1}(\tilde{Q}) = 0$ and $F = T$. Then from the digram we have

$$\mu_{t_0}(T) = \mu_{t_0}(E) = \mu_{t_0}(\tilde{Q})) > \mu_{t_0}(L^{n-1} \otimes \mathcal{I}_Y).$$

Which contradicts the stability of E at t_0.

Hence the only destabilizer of $\mathrm{ch} = (1, \ell, n^2 d + 2d(n-1) - 1)$ is given by $L^{n-1} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y$ with $|Y| \leq \left\lfloor \dfrac{(n-1)(d(n-2)-1)}{n} \right\rfloor$. □

Chapter 6

Minimal Degree Torsion Sheaves on Abelian Surfaces

Let S be an abelian surface and L an ample line bundle on S with $c_1(L) = \ell$ and $\ell^2 = 2d$. In this chapter we will prove some properties of the moduli space $\mathcal{M}_{(0,\ell,\beta)}$. Then we will locate the walls of this Chern character for $0 < \beta \leq 2d$ and give examples for some values of β.

6.1 Torsion sheaves with $\mathrm{ch} = (0, \ell, \beta)$

In Chapter 5 we proved that the moduli space $\mathcal{M}_{(0,\ell,\beta)}$ is IT_0 at $s = 0$ if and only if $\beta > d$ (see Lemma 5.2.6). We will show that for $\beta \leq 0$, $\mathcal{M}_{(0,\ell,\beta)}$ is IT_1 in the following lemma

Lemma 6.1.1. *Let T be a torsion sheaf with $\mathrm{ch}(T) = (0, \ell, \beta)$, then $\mathcal{M}_{(0,\ell,\beta)}$ is WIT_1 if $\beta \leq 0$.*

Proof. We claim to prove that $\mathrm{ch}(T) = (0, \ell, 0)$ is WIT_1, but first we prove that the statement is true for $\beta \leq 0$. For $\beta = -2d$, there exist $\hat{x}, \hat{y} \in \hat{S}$ and a short exact sequence

$$0 \to L^{-1} P_{\hat{x}} \to P_{\hat{y}} \to T \to 0, \tag{6.1.1}$$

where T has Chern character $\mathrm{ch}(T) = (0, \ell, -2d)$. Since $L^{-1}P_{\hat{x}}$ is IT_2 and $P_{\hat{y}}$ is WIT_2. Now take the cohomology of (6.1.1) we obtain

$$0 \to \Phi^1(T) \to \Phi^2(L^{-1}P_{\hat{x}}) \xrightarrow{f} \Phi^2(P_{\hat{y}}) \to \Phi^2(T) \to 0. \tag{6.1.2}$$

Since $f \neq 0$, then f is surjective and $\Phi^2(T) = 0$ so T is WIT_1 and $\Phi^1(T)$ is torsion free. We can use induction to prove that the statement is true for all $\beta < -2d$.

To prove that the statement is true for $-2d \leq \beta \leq 0$, supposeT' with $\mathrm{ch}(T') = (0, \ell, -n)$ is WIT_1 for $0 < n < 2d$. Take $x \in \mathrm{Supp}(T)$, then we have a short exact sequence

$$0 \to T \to T' \to \mathcal{O}_x \to 0 \tag{6.1.3}$$

where $\mathrm{ch}(T) = (0, \ell, -n+1)$. Taking the cohomology of (6.1.3) we get a long exact sequence

$$0 \to \mathcal{P}_x \to \Phi^1(T) \to \Phi^1(T') \to 0, \tag{6.1.4}$$

where $\mathrm{ch}(\Phi^1(T')) = (n, \ell, 0)$. We know that there are no walls at $s = 0$, then $\Phi^1(T)$ is Gieseker stable sheaf and so $\mathrm{ch}(\Phi^1(T')) = (n, \ell, 0)$ is torsion free. Hence T is WIT_1 for $0 < n < 2d$.

For $\beta = 0$, the same argument still shows that T with $\mathrm{ch}(T) = (0, \ell, 0)$ is WIT_1 in the same way. But in this case $\Phi^1(T)$ is Simpson stable. □

Lemma 6.1.2. *If $\frac{d}{2} < \beta \leq d$ and T is pure torsion with* $\mathrm{ch}(T) = (0, \ell, \beta)$ *then* $|\Phi^1(T)| = 1$.

Proof. Since there are no walls for $s = 0$ (Lemma 5.2.2), then $\Phi(T)$ is Bridgeland stable for all $t > 0$ which means $\Phi^0(T)$ is Gieseker semistable and $\Phi^1(T)$ is skyscraper. Therefore if $|\Phi^1(T)| > 1$, then $\chi(\Phi^0(T)) > 1$ which contradicts Bogomolov inequality for $\beta > \frac{d}{2}$ as $r(\Phi^0(T)) = \beta$. Hence $|\Phi^1(T)| \leq 1$. But $|\Phi^1(T)| = 0$ implies that T is WIT_0 which contradicts Lemma 5.2.6. Hence $|\Phi^1(T)| = 1$. □

Let T be a torsion sheaf with $\mathrm{ch}(T) = (0, \ell, \left\lfloor \frac{d}{2} \right\rfloor)$. The following theorem shows that Lemma 6.1.2 fails if $\beta = \left\lfloor \frac{d}{2} \right\rfloor$:

Theorem 6.1.3. *Let E be a μ-stable vector bundle with $\mathrm{ch}(E) = (2, \ell, \left\lfloor \frac{d}{2} \right\rfloor)$, then:*

1. *There exists T with $\mathrm{ch}(T) = (0, \ell, \left\lfloor \frac{d}{2} \right\rfloor)$ such that $|\Phi^1(T)| = 2$.*
2. *There exist a 0-dimensional subscheme X with length $|X| = \left\lceil \frac{d}{2} \right\rceil$ such that $\Phi^1(L\mathcal{I}_X) \neq 0$.*

Proof. The existence of a μ-stable vector bundle E with the given Chern character is discussed in detail in [Yos01]. Take E to be as in the theorem, then E has a section $\mathcal{O} \to E \to Q$ where $Q = E/\mathcal{O}$ is torsion-free sheaf. Furthermore, since E has a positive Euler number, then there is a nonzero map $\mathcal{P}_{\hat{x}} \to E$ for all $\hat{x}$ and $\tilde{Q} = E/\mathcal{P}_{\hat{x}}$ is torsion-free sheaf $\tilde{Q} = L\mathcal{P}_{-\hat{x}}\mathcal{I}_{X'}$. Then we get the following diagram,

$$\begin{tikzcd}
H & \mathcal{P}_{\hat{x}} \arrow[l] \arrow[d] \arrow[r, equal] & \mathcal{P}_{\hat{x}} \arrow[d] \\
\mathcal{O} \arrow[u] \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[dr] & L\mathcal{I}_X \arrow[d] \\
\mathcal{O} \arrow[r] & L\mathcal{P}_{-\hat{x}}\mathcal{I}_{X'} \arrow[r] & T
\arrow[from=1-1, to=2-2]
\end{tikzcd}$$

where $|X| = \left\lceil \frac{d}{2} \right\rceil$, $\mathrm{ch}(T) = (0, \ell, \left\lfloor \frac{d}{2} \right\rfloor)$ and $H = \ker(E \to T)$. From the diagram, $\mathrm{Ext}^1(T, \mathcal{O})$ and $\mathrm{Ext}^1(T, \mathcal{P}_{\hat{x}})$ are not zero and $\chi(T) = \left\lfloor \frac{d}{2} \right\rfloor$, then $\Phi^1(T)_0 \neq 0$ and $\Phi^1(T)_{\hat{x}} \neq 0$. If the sequence $\mathcal{O} \to E \to L\mathcal{I}_X$ has a non trivial extension, then $|\Phi^1(L\mathcal{I}_X)| \geq 1$ which proves (2). $\square$

Similarly for an integer $r \geq 2$, if $\left\lfloor \frac{d}{r+1} \right\rfloor < \beta \leq \left\lfloor \frac{d}{r} \right\rfloor$, there is T with $\mathrm{ch}(T) = (0, \ell, \beta)$ such that $|\Phi^1(T)| = r$. On the other hand, these T with $\mathrm{ch}(T) = (0, \ell, \beta)$ can arise as Bridgeland destabilizing torsion-free sheaves in the following way:

Proposition 6.1.4. *For any $\chi > 0$ and $r > 0$ such that $d \geq r\chi > 0$, let A be a μ-stable locally free sheaf with Chern character $\mathrm{ch}(A) = (r, \ell, \chi)$. Then there is a short exact sequence*

$$0 \to H_X \to A \to T \to 0 \tag{6.1.5}$$

where $|X| = r$, T *is pure and* $\chi(T) = \chi$.

Proof. Pick μ-stable A with Chern character $\mathrm{ch}(A) = (r, \ell, \chi)$, then $H^0(A \otimes \mathcal{P}_{\hat{x}}) \neq 0$ for all $\hat{x}$. Pick r distant $\mathcal{P}_{\hat{x}_1}, ..., \mathcal{P}_{\hat{x}_r}$ and injective maps $\mathcal{P}_{\hat{x}_i} \hookrightarrow A$ for $i = 1, \ldots, r$. Then there is a map $f : \mathcal{P}_{\hat{x}_1} \oplus ..., \oplus \mathcal{P}_{\hat{x}_r} \to A$. Let $K = \mathrm{Ker}(f)$ with $\mathrm{ch}(K) = (k, 0, -\alpha)$and f factors through Q with $\mathrm{ch}(Q) = (r - k, c_1\ell, \alpha)$, then we obtain the following diagram:

$$\begin{tikzcd}
 & & \mathcal{P}_{\hat{x}_i} \arrow[dl] \arrow[dr] & \\
K \arrow[r] & \mathcal{P}_{\hat{x}_1} \oplus \ldots \oplus \mathcal{P}_{\hat{x}_r} \arrow[rr] \arrow[dr] & & A \\
 & & Q \arrow[ur] &
\end{tikzcd}$$

where $i = 1, \ldots, r$. Now we have tow cases, if $r(Q) = r(A)$, then $K = 0$. The Second case if $r(Q) < r(A)$, since A is stable ,then $\mu(Q) < \mu(A)$, then $\deg(Q) \leq 0$ and so $\deg(Q) = 0$. By Bogomolov inequality for Q and K, we get $\alpha = 0$. Hence $K = 0$ and $Q = \bigoplus_{i=1}^{r} \mathcal{P}_{\hat{x}_i}$. Hence there is a short exact sequence

$$0 \to H_X \to A \to T \to 0,$$

where $X = \{\hat{x}_1, ..., \hat{x}_r\}$. □

The following allows us to explicitly construct destabilizing objects for our torsion sheaves

Proposition 6.1.5. *For any positive integers* β, α *and* γ *such that* $0 \leq \beta \leq 2d$, $\alpha + 2d - \beta \leq \gamma$, *there is a pure torsion sheaf* T *with* $\mathrm{ch}(T) = (0, \ell, \beta)$ *and* $L\mathcal{I}_X$ *with* $|X| = \gamma$, *and a long exact sequence*

$$0 \to \mathcal{I}_Y \to L\mathcal{I}_X \to T \to \mathcal{O}_{Y'} \to 0 \tag{6.1.6}$$

with $|Y'| = \alpha$ *and* $|Y| = \beta + \gamma - \alpha - 2d$.

Proof. Pick D_L and a discrete Y such that $Y \cap D_L = \phi$. Pick $Z \subset D_L$, where $X = Y \cup Z$ such that $|X| = d - \chi$ and $|Z| = d - \beta + \alpha$. Then there is an injective map $f : \mathcal{I}_Y \hookrightarrow L\mathcal{I}_X$

and $\mathrm{Cok}(f) := T'$. For each $y \in Y$, f does not factor through $\mathcal{I}_{Y\setminus\{y\}} \hookrightarrow L\mathcal{I}_X$ then T' is pure.

Now pick a discrete $Y' \subset D_L$ and non trivial extension ϵ:

$$0 \to T' \to T \to \mathcal{O}_{Y'} \to 0 \tag{6.1.7}$$

such that for each $y \in Y'$, $0 \neq \mathrm{Im}(\epsilon) \in \mathrm{Ext}^1(\mathcal{O}_y, T')$. Take the composition of $L\mathcal{I}_X \to T' \to T$ then we have the short exact sequence $0 \to \mathcal{I}_Y \to L\mathcal{I}_X \to T \to \mathcal{O}_{Y'} \to 0$. □

6.2 Computing the Walls

In this section we will identify the walls of the Chern character $\mathrm{ch} = (0, \ell, \beta)$ where $0 \leq \beta \leq d$ and find all possible destabilizers of this given ch. In the previous section we showed that there are no walls for the Chern character $\mathrm{ch} = (r, \ell, \chi)$ when $s = 0$ for all $r \geq 0$ and $\chi > 0$. But if we vary s, then we can find walls for ch. The slope function of $A \in \mathcal{A}_s$ with $\mathrm{ch}(A) = (r, c\ell, \chi)$ is given by

$$\mu_{s,t}(A) = \frac{\chi - 2dcs - dr(t^2 - s^2)}{2td(c - rs)} \tag{6.2.1}$$

Let T be a sheaf with $\mathrm{ch}(T) = (0, \ell, \beta)$ then the slope function of T is given by

$$\mu_{s,t}(T) = \frac{\beta - 2ds}{2td} \tag{6.2.2}$$

Suppose $F \in \mathcal{A}_s$ with $\mathrm{ch}(F) = (r, c\ell\chi)$ destabilizes T. Then we have a short exact sequence

$$0 \to F \to T \to Q \to 0$$

in $\mathcal{A}_s$. Taking cohomology we see that $H^{-1}(F) = 0$. Then $F \in T_s$. Also $H^{-1}(Q) \in F_s$ is torsion-free, and since $0 \to H^{-1}(Q) \to F \to T$ is exact, F is torsion-free.

We also have

$$\mu_{s,t}(F) - \mu_{s,t}(T) \geq 0.$$

Therefore, from (1.5.9), the destabilizing condition of T is

$$\chi - c\beta - r(ds^2 - s\beta) - drt^2 \geq 0, \tag{6.2.3}$$

and the walls are given by

$$\chi - c\beta - r(ds^2 - s\beta) - drt^2 = 0 \tag{6.2.4}$$

which are semicircles with centre on s-axis is $C = \frac{\beta}{2d}$.

Definition 6.2.1. *If T is a torsion with* $\mathrm{ch}(T) = (0, \ell, \beta)$*, then we denote the number of walls of T by $W(\beta)$.*

Definition 6.2.2. *If T is a torsion with* $\mathrm{ch}(T) = (0, \ell, \beta)$ *and A with* $\mathrm{ch}(A) = (r, n\ell\chi)$ *destabilizes T, then we denote the number of walls of T by $W(\beta, r, n)$.*

Now we will prove that for every wall for T with $\chi(T) = \beta$ there is corresponding wall for $\chi(T) = 2d - \beta$ and $-\beta$ in the following lemmas

Lemma 6.2.3. *If W is a wall for* $\mathrm{ch} = (0, \ell, \beta)$*, then there exists a W' wall for* $\mathrm{ch} = (0, \ell, \beta + 2d)$ *such that* $W' = W + (2d, 0)$.

Proof. If W is a wall for $\mathrm{ch} = (0, \ell, \beta)$, then there exist $T \in \mathcal{M}^{BS}_{\mathrm{ch}}$ and a short exact sequence

$$0 \to A \xrightarrow{f} T \xrightarrow{g} Q \to 0 \tag{6.2.5}$$

such that $\mu_{s_0,t_0}(A) - \mu_{s_0,t_0}(T) = 0$ and $(s_0, t_0) \in W$. Tensor (6.2.5) by L we get

$$0 \to A \otimes L \xrightarrow{f \otimes \mathrm{id}} T \otimes L \xrightarrow{g \otimes \mathrm{id}} Q \otimes L \to 0 \tag{6.2.6}$$

Since L is flat then the exactness of (6.2.6) follows from the exactness of (6.2.5). Now take $(s,t) \in W'$ where $s = s_0 + 2d$ and $t = t_0$. Then

$$\begin{aligned}
&\mu_{s_0+2d,t_0}(\hat{A}) - \mu_{s_0+2d,t_0}(\hat{T}) \\
&\qquad = \frac{-\chi + 2d(s_0+2d) + dr(t_0^2 - (s_0+2d)^2)}{2t_0 d(c + r(s_0+2d))} \\
&\qquad + \frac{(1 + r(s_0+2d))(\beta - 2d(s_0+2d))}{2t_0 d(c + r(s_0+2d))} \\
&\qquad = \frac{-\chi + \beta + r(ds_0^2 - s_0\beta) + drt_0^2]}{2t_0 d(c + r(s_0+2d))}
\end{aligned}$$

From (6.2.4), $\mu_{s,t}(\hat{A}) - \mu_{s,t}(\hat{T}) = 0$, then $(s_0+2d, t_0) \in W'$. □

In particular, if W is a wall for $\mathrm{ch} = (0,\ell,\beta)$, then there exists a W' wall for $\mathrm{ch} = (0,\ell,2d-\beta)$ such that $W' = (2d,0) - W$. The generalization of Lemma 6.2.3 for $\chi(T) = 2Nd + \beta$ for $N > 1$ follows in the same way by tensoring (6.2.5) by L^N.

Lemma 6.2.4. *If W is a wall for $\mathrm{ch} = (0,\ell,\beta)$, then there exists a W' wall for $\mathrm{ch} = (0,\ell,-\beta)$ such that $W' = -W$.*

Proof. Take the dual of (6.2.5) and we want to show that $\mu_{s,t}(\hat{A}) - \mu_{s,t}(\hat{T}) = 0$ where $(s,t) \in W'$, $s = -s_0$ and $t = t_0$. Then

$$\begin{aligned}
\mu_{-s_0,t_0}(\hat{A}) - \mu_{-s_0,t_0}(\hat{T}) &= \frac{-\chi + 2ds_0 + dr(t_0^2 - s_0^2) + (1 + rs_0)(\beta - 2ds_0)}{2t_0 d(c + rs_0)} \\
&= \frac{-\chi + \beta + r(ds_0^2 - s_0\beta) + drt_0^2}{2t_0 d(c + rs)}
\end{aligned}$$

From (6.2.4), $\mu_{s,t}(\hat{A}) - \mu_{s,t}(\hat{T}) = 0$, then $(-s_0, t_0) \in W'$. □

Previous lemmas gives the following proposition:

Proposition 6.2.5. *If T is a torsion with $\mathrm{ch}(T) = (0,\ell,\chi(T))$, then*

1. *$W(2d-\beta) = W(\beta)$,*

2. *$W(\beta + 2Nd) = W(\beta)$, for $N \geq 1$,*

3. *$W(-\beta) = W(\beta)$.*

Therefore, it is enough to find walls for the Chern character $\mathrm{ch} = (1, \ell, \beta)$ where $0 \le \beta < 2d$ and $0 < s \le \frac{1}{2}$.

Now we want to identify all walls for $\mathrm{ch}(T) = (0, \ell, \beta)$ where $0 \le \beta < 2d$. For $s > 0$, choose a Bridgeland stable sheaf A with $\mathrm{ch}(A) = (r, c\ell, \chi)$ such that A destabilizes T, then we have a short exact sequence

$$0 \to A \to T \to B \to 0$$

Taking the cohomology we obtain

$$0 \to H^{-1}(B) \to A \to T \to H^0(B) \to 0 \tag{6.2.7}$$

where $H^0(B)$ with $\chi(H^0(B)) = \alpha$ and $H^{-1}(B) \in F_s$ with $\chi(H^{-1}(B)) = \chi - \beta - \alpha$. If $A \to T$ factors through T' then $c_1(T') = \ell$ and $c_1(H^{-1}(B)) = (c-1)\ell$. On the other hand $A \in \mathcal{T}_s$ then $\mu(A) > 2ds$ and hence $r < cs$.

On the other hand, Maciocia in [Mac14] proved that the line $s = C_0 = \dfrac{\chi(T)}{\ell^2.c_1(T)} = \dfrac{\beta}{2d}$ must intersect all the walls of T. Therefore we will fix $s = \frac{\beta}{2d}$ to get the first Chern character $c_1(A)$ of a destabilizer A of T in the following Proposition

Proposition 6.2.6. *If T is a torsion with* $\mathrm{ch}(T) = (0, \ell, \beta)$ *and A destabilizes T in $\mathcal{A}_{\frac{\beta}{2d}}$, then the first Chern class $\tilde{c}_1(A)\ell$ of A is given by*

$$\tilde{c}_1(A) = \left\lfloor \frac{r\beta}{2d} \right\rfloor + 1, \tag{6.2.8}$$

where $r = r(A)$.

Proof. As $A \in \mathcal{T}_{\frac{\beta}{2d}}$, then $\mu(A) > \frac{\beta}{2d}$ and so

$$\tilde{c}_1(A) > \frac{r\beta}{2d} \tag{6.2.9}$$

On the other hand, since A is Bridgeland stable then it is simple and so satisfies

Bogomolov (Lemma 2.5.8). Hence $\chi \leq \frac{cd}{r}$. Also $H^{-1}(G) \in \mathcal{F}_{\frac{\beta}{2d}}$, then $\frac{c-1}{r} \leq \frac{\beta}{2d}$ and so

$$\tilde{c}_1(A) \leq \frac{r\beta}{2d} + 1 \tag{6.2.10}$$

Therefore $c_1(A)$ can be written in terms of $r(A), \beta$ and d as

$$\tilde{c}_1(A) = \left\lfloor \frac{r\beta}{2d} \right\rfloor + 1. \tag{6.2.11}$$

□

Remark 6.2.7. *The radii of the walls are given by $R_0 = \sqrt{\frac{\chi - c\beta}{dr} + \frac{\beta^2}{4d^2}}$ are bounded above by $\frac{\beta}{2d}$ and we observe that the biggest wall which has the maximum radius is reached when $\chi(A) = \tilde{c}_1(A)\beta$, and so $\chi(A) \leq \tilde{c}_1(A)\beta$.*

Example 6.2.8. *If $c_1(A) = \ell$, then for $0 \leq \beta \leq 2d$, we get $r < \frac{2d}{\beta}$ and $c_1(H^{-1}(B)) = 0$.*

The following Lemma proves the semistability of $H^{-1}(B)$ in this case:

Lemma 6.2.9. *If $c_1(A) = \ell$, then $H^{-1}(B)$ is semistable.*

Proof. Suppose that $D \hookrightarrow H^{-1}(B)$ is destabilizing $H^{-1}(B)$, then $\mu(D) \leq \frac{\beta}{2d}2d = \beta$ and $\deg(D) > 0$. Then $r(D) \geq \frac{\deg(D)}{\beta} \geq \frac{2d}{\beta}$, but $r(D) < r < \frac{2d}{\beta}$ as $\mu(A) > \beta$ which is contadiction. □

Therefor $H^{-1}(B)$ is semistable and by applying Bogololov's inequality we get $\chi \leq \beta - \alpha$. Hence $\chi(A)$ is bounded by

$$\beta - \frac{r\beta^2}{4d} < \chi(A) \leq \min\left(\frac{d}{r}, \beta\right). \tag{6.2.12}$$

Thus if $c_1(A) = \ell$, the number of walls $W(r, \beta)$ is given by

$$W(\beta, r, 1) = \begin{cases} \left\lceil \frac{\beta^2 r}{4d} \right\rceil, & 0 < \beta \leq \frac{d}{r} \\ \left\lceil \frac{d}{r} \right\rceil + \left\lceil \frac{\beta^2 r}{4d} \right\rceil - \beta, & \frac{d}{r} < \beta < \frac{2d}{r} \end{cases} \tag{6.2.13}$$

Lemma 6.2.10. *For the biggest wall (when $\chi = c\beta$), we have $c_1(A) = \ell$.*

Proof. From (6.2.8) we have:

$$\tilde{c}_1(A) = \left\lfloor \frac{r\beta}{2d} \right\rfloor + 1$$

Let $\gamma = \left\lfloor \frac{r\beta}{2d} \right\rfloor$, then $\tilde{c}_1(A) = \gamma + 1$ and there are unique $\alpha, \gamma \in \mathbb{Z}$ such that

$$r = \frac{2d}{\beta}\gamma + \alpha, \qquad \text{where } 0 \le \alpha < \frac{2d}{\beta}. \tag{6.2.14}$$

Since A is simple and from Proposition 2.5.8 it satisfies Bogomolov's inequality, then

$$\beta \cdot r(A) \le \tilde{c}_1(A) \cdot d.$$

Therefore,

$$2d\gamma + \beta\alpha \le (\gamma + 1)d.$$

Then we have $0 \le \beta\alpha \le (1-\gamma)d$, so $0 \le \gamma < 1$. Hence $c_1(A) = \ell$ when $\chi = \tilde{c}_1(A) \cdot \beta$.

□

To get a general formula for the number of walls of T with $\mathrm{ch}(T) = (0, \ell, \beta)$ we need to bound the Euler characteristic of A.

Proposition 6.2.11. *If A with $\mathrm{ch}(A) = (r, n\ell, \chi(A))$, then $\chi(A)$ is bounded by:*

$$n\beta - \frac{\beta^2 r}{4d} < \chi(A) \le \min\left(\frac{n^2 d}{r}, n\beta\right) \tag{6.2.15}$$

Proof. From (6.2.8) we have

$$\frac{2d(n-1)}{r} < \beta < \frac{2dn}{r}, \tag{6.2.16}$$

and in this range $R_o^2 > 0$ if and only if

$$\chi > n\beta - \frac{\beta^2 r}{4d}. \tag{6.2.17}$$

Since A is semistable, then

$$\chi(A) \le \frac{n^2 d}{r}. \tag{6.2.18}$$

From Remark 6.2.7, $\chi \le n\beta$, then $n\beta - \frac{\beta^2 r}{4d} < \chi(A) \le \min\left(\frac{n^2 d}{r}, n\beta\right)$. □

We observe that $\frac{n^2 d}{r} < n\beta$ unless $n = 1$. Therefore if we fix an integer β is in $\left(\frac{2d(n-1)}{r}, \frac{2dn}{r}\right]$, then the number of walls is given by the following function:

$$W(\beta, r, n) = \left\lceil \frac{n^2 d}{r} \right\rceil + \left\lceil \frac{\beta^2 r}{4d} \right\rceil - n\beta \tag{6.2.19}$$

As we showed early in this chapter, $r(A) < \frac{2dn}{\beta}$. Let $r_\beta = \left\lceil \frac{2dn}{\beta} \right\rceil - 1$ and $r'_\beta = \left\lfloor \frac{2d(n-1)}{\beta} \right\rfloor$, then the number of all possible walls of T when β is in $\left(\frac{2d(n-1)}{r}, \frac{2dn}{r}\right]$ and $n > 1$ is given by:

$$\begin{aligned} W(\beta, n) &= \sum_{r'_\beta}^{r_\beta} \left[\left\lceil \frac{n^2 d}{r} \right\rceil + \left\lceil \frac{\beta^2 r}{4d} \right\rceil - n\beta \right] \\ &= \sum_{r'_\beta}^{r_\beta} \left\lceil \frac{n^2 d}{r} \right\rceil + \sum_{r=1}^{r_\beta} \left\lceil \frac{\beta^2 r}{4d} \right\rceil - n\beta r_\beta. \end{aligned}$$

6.3 Fourier-Mukai Transforms for $\mathrm{ch} = (0, \ell, \beta)$

In this section we will give a Fourier-Mukai transform Φ_1 which is acting on the line $s = \frac{\beta}{2d}$ and given by the cohomology transform associated with $\mathbb{F} \in \mathcal{D}(S \times \tilde{S})$.

Mukai in [Muk78] gave the condition of the existence of a universal sheaf $\mathbb{F}$ (see [Muk87, Theorem A.6]), if the greatest common divisor $\gcd(r(\mathbb{F}), \chi(\mathbb{F})) = 1$ then there is a universal sheaf on $S \times \tilde{S}$. Therefore , we will take $c = 1$ in (2.2.1) and the greatest common divisor of $r(\mathbb{F}), c_1(\mathbb{F})$ and $\chi(\mathbb{F})$ is 1. Then we have $\mathrm{ch}(\mathbb{F}|_S) = (ax^2, -xy\ell, by^2)$, $\mathrm{ch}(\mathbb{F}|_{\tilde{S}}) = (by^2, -yz\ell, az^2)$ where $\ell^2 = 2ab = 2d$ and $\gcd(ax, by) = 1$. We use the notation $n||m$ to mean $n|m$ and $\gcd(n, \frac{m}{n}) = 1$ and it is called "**exactly devides**".

Now consider Chern character of the form $(0, \ell, \beta)$. We will give a Fourier-Mukai Transform for $\mathrm{ch} = (0, \ell, \beta)$ in two cases:

I. If β is odd: In this case, we are interested in the case that $\gcd(\beta, d)||d$. Consider $\beta = ax$, and take $a = \gcd(\beta, d)$, then $\gcd(\frac{d}{a}, a) = 1$. The Fourier-Mukai transformation matrix is

$$\begin{pmatrix} ax & by \\ w & z \end{pmatrix},$$

take $y = 2$ and from (2.2.1) we get

$$\begin{pmatrix} ax^2 & -2x\ell & 4b \\ -xw\ell & axz+2bw & -2z\ell \\ bw^2 & -2wz\ell & az^2 \end{pmatrix} \tag{6.3.1}$$

where $axz - 2bw = 1$. We also have $\gcd(\beta, a) = a$ and $\gcd(x, b) = \gcd(x, 2) = 1$. Then

$$\begin{aligned} \mathrm{ch}(\Phi_1((r, c_1\ell, \chi))) =& (ax^2r - 4dxc_1 + 4b\chi, (-xwr + 4bwc_1 + c_1 - 2\chi z)\ell, \\ & brw^2 - 4dc_1wz + a\chi z^2). \end{aligned}$$

II. If β is even. We are interested in the case that $\gcd(\frac{\beta}{2}, d)||d$. Consider $\beta = 2ax$, and take $a = \gcd(\frac{\beta}{2}, d)$, then $\gcd(\frac{d}{a}, a) = 1$. To find the Fourier-Mukai transformation matrix, take $y = 1$and from (2.2.1) the Fourier-Mukai transformation Φ_2 exists and it is given by

$$\begin{pmatrix} ax^2 & -x\ell & b \\ -xw\ell & axz+bw & -z\ell \\ bw^2 & -wz\ell & az^2 \end{pmatrix} \tag{6.3.2}$$

where $axz - bw = 1$. We also have $\gcd(\beta, d) = a$ and $\gcd(x, b) = \gcd(x, 2) = 1$.

Then

$$\begin{aligned}\mathrm{ch}(\Phi_1((r, c_1\ell, \chi))) = &(ax^2r - 2dxc_1 + b\chi, (-xwr + 2c_1axz - c_1 - \chi z)\ell,\\ &brw^2 - 2dc_1wz + a\chi z^2)\end{aligned}$$

Proposition 6.3.1. *In the case where* $\gcd(\beta, d)||d$*, if* Φ_1 *is the Fourier-Mukai transform* $s = \frac{d}{2\beta}$*, then there is* $0 < \beta' \leq d$ *such that* $\mathrm{ch}(\Phi_1(T)) = (0, \ell, \beta')$*.*

Proof. If $a \in \mathbb{Z}$ such that $\gcd(d, \beta) = a$, $\beta = ax$ and $ab = d$ where a, b are coprime we obtain $z = \frac{1+2bw}{\beta}$. If β is odd, applying a weak Fourier-Mukai transform (6.3.1) we get

$$\mathrm{ch}(\Delta\Phi_1((0, \ell, \beta))) = (0, \ell, az).$$

Hence $\beta' = az$ where z is a solution of $axz - 2bw = 1$. Then $0 \leq az < 2a \leq 2d$. Similarly, if β is even then applying the Fourier-Mukai transform (6.3.2) we get

$$\mathrm{ch}(\Delta\Phi_1((0, \ell, \beta))) = (0, \ell, 2az).$$

Then $\beta' = 2az$ where z is a solution of $axz - bw = 1$. Then $0 \leq az < a \leq d$. □

Corollary 6.3.2. $W(\beta) = W(\beta')$.

Proposition 6.3.3. *If* β *is odd, the Fourier-Mukai transforms which are given in (6.3.1) takes the stability condition* $\sigma_{(\frac{\beta}{2d}, t)}$ *to* $\sigma_{(s', t')}$ *where* $s' = zs/x$ *and* $t' = 1/4bdt$*. If* β *is even, the Fourier-Mukai transforms which are given in (6.3.2) takes the stability condition* $\sigma_{(\frac{\beta}{2d}, t)}$ *to* $\sigma_{(s', t')}$ *where* $s' = zs/x$ *and* $t' = 1/4bdt$*.*

Proof. Let $k = \beta/2d + it = x/2b + it$. Since $\mathrm{ch}(\Phi_1(L^k)) = \lambda\,\mathrm{ch}(L^{k'})$ where $k' = s' + it'$.

Therefore from (6.3.1), we have

$$\begin{aligned}
\operatorname{ch}\big(\Phi_1((1,k\ell,k^2d))\big) =& \big(ax^2-4dxk+4bk^2d,(-xw+kaxz-2bwk-2k^2zd)\ell,\\
& bw^2-4dkwz+ak^2z^2d\big)\\
=& \big(a(x-2bk)^2,-(x-2bk)(w-azk)\ell,b(w+akz)^2\big)\\
=& a(x-2bk)^2\big(1,-\frac{w-azk}{a(x-2bk)}\ell,\frac{b(w+akz)^2}{a(x-2bk)^2}\big)
\end{aligned}$$

Therefore, $\lambda = a(x-2bk)^2$ and

$$\begin{aligned}
k' =& -\frac{w-azk}{a(x-2bk)}\\
=& \frac{axz-2bw-2dzti}{-4bdti}\\
=& \frac{az}{2d}+\frac{1}{4bdt}i.
\end{aligned}$$

Hence $s' = az/2d = zs/x$ and $t' = 1/4bdt$. □

Definition 6.3.4. *We denote the radius of the maximum wall (the minimum wall) by R_{max} (R_{min}) respectively.*

Remark 6.3.5. *When β is odd, Fourier-Mukai transform (6.3.1) takes t to $t' = 1/4dbt$. Then we get*

$$R_{min} = \frac{1}{4dbR_{max}}. \tag{6.3.3}$$

Since $R_{max} = \beta/2d$, then $R_{min} = 1/2b\beta$.

6.4 Examples

As the previous calculations showed that we can not give a general picture of the walls when $1 \le \beta \le 2d$, then we will study particular values of β

6.4.1 $\beta = 0$

If $\beta = 0$, then $s = 0$ and so $R_0 = 0$. Hence there are no walls for $\mathrm{ch} = (0, \ell, 0)$ which is the case that we proved in Lemma 5.2.2.

6.4.2 $\beta = 1$

Recall that for the biggest wall we have $c_1(A) = \ell$ (See Lemma 6.2.10). Now fix $s = \frac{1}{2d}$, the destabilizing condition is given by

$$\chi - 1 + \frac{r}{4d} > 0. \tag{6.4.1}$$

Proposition 6.4.1. *If T is a torsion with* $\mathrm{ch}(T) = (0, \ell, 1)$, *then* $W(1) = d$.

Proof. From (6.2.12) we have $1 \geq \chi > 1 - \frac{r}{4d} > 0$ and $0 \leq \alpha < \frac{r}{d} = \frac{1}{2}$ so $\alpha = 0$.The centre of the biggest wall is $C_0 = \frac{1}{2d}$ and the radius is $R_0 = \frac{1}{2d}$. From semistability of $H^{-1}(B)$ we have $\chi \leq 1$. Hence $\chi = 1$ and $0 < r \leq d$. Therefore at $\beta = 1$, χ is fixed and the rank of destabilizers varies between $0 < r \leq d$. □

These correspond to $|\Phi^1(T)| \leq d$ from Lemma 6.1.3. On the other hand, to get the matrix of the Fourier-Mukai transform Φ_1 which is acting on the line $s = \frac{1}{2d}$, take $b = d$, $w = 0$ and $a = x = z = 1$ in (6.3.1). Then the Fourier-Mukai transform is given by

$$\begin{pmatrix} 1 & -2\ell & 4d \\ 0 & 1 & -2\ell \\ 0 & 0 & 1 \end{pmatrix} \tag{6.4.2}$$

and $\mathrm{ch}(\Phi_1[1]((0, \ell, 1)) = (0, \ell, 1)$.

6.4.3 $\beta = 2$

Fix $s = \frac{1}{d}$, the destabilizing condition is given by

$$\chi - 2 + \frac{r}{d} > 0. \tag{6.4.3}$$

Proposition 6.4.2. *If T is a torsion with* $\mathrm{ch}(T) = (0, \ell, 2)$*, then* $W(2) = \left\lfloor \frac{d}{2} \right\rfloor$.

Proof. Similar to the previous case, when $\beta = 2$ then $\chi = 2$ and there is only one multiple wall with center $C_0 = \frac{1}{d}$ and radius $R_0 = \frac{1}{d}$. Moreover the rank of destabilizers varies between $0 < r \leq \frac{d}{2}$. Hence $W(2) = \left\lfloor \frac{d}{2} \right\rfloor$. □

To get the matrix of a Fourier-Mukai transform Φ_2 when $\beta = 2$ who is acting on the line $s = \frac{1}{d}$, take $b = d$, $w = 0$ and $a = x = z = 1$ in (6.3.2). Then the Fourier-Mukai transformation is given by

$$\begin{pmatrix} 1 & -\ell & d \\ 0 & 1 & -\ell \\ 0 & 0 & 1 \end{pmatrix} \tag{6.4.4}$$

and $\mathrm{ch}(\Phi_1[1]((0, \ell, 2)) = (0, \ell, -2)$

6.4.4 $\beta = 3$

Proposition 6.4.3. *If T is a torsion with* $\mathrm{ch}(T) = (0, \ell, 3)$*, then*

$$W(r,3,1) = \begin{cases} \left\lceil \dfrac{d}{18} \right\rceil + 1 \ \textit{if } d \textit{ is even} \\ \left\lceil \dfrac{d+9}{18} \right\rceil \ \textit{if } d \textit{ is odd} \end{cases} \tag{6.4.5}$$

Proof. Fix $s = \frac{3}{2d}$. From (6.2.12), $\chi = 2, 3$. If $\chi = 3$, we get the biggest multiple wall with center $C_0 = \frac{3}{2d}$ and radius $R_0 = \frac{3}{2d}$. If $\chi = 2$, the radius is given by $R_0 = \sqrt{\dfrac{-1}{rd} + \dfrac{9}{4d^2}}$. Let d be an even integer and from Bogomolov for A, the rank of destabilizers varies between $0 < r < \frac{d}{2}$. As the relation between the radius and the rank is a positive relation then the greatest radius is given when $r = \frac{d}{2}$. Therefore $R_0 = \frac{1}{2d}$.

If $r = \frac{d-2}{2}$ then $R_0 = \sqrt{\dfrac{-2}{(d-2)d} + \dfrac{9}{4d^2}}$ and so$R_0^2 > 0$ if and only if $d > 18$. Hence there are two walls when $d \leq 18$. If we repeat the same calculations when $r = \frac{d-4}{2}$ we find that the number of walls is 3 when $18 < d \leq 36$. Generally if d is even, then the number of walls is

$$W(r,3,1) = \left\lceil \frac{d}{18} \right\rceil + 1.$$

Similarly if d is odd, then $0 < r < \frac{d-1}{2}$. If $r = \frac{d-1}{2}$, then $R_0^2 > 0$ if and only if $d > 9$. Thus the number of walls when d is odd is

$$W(r,3,1) = \left\lceil \frac{d+9}{18} \right\rceil$$

□

To get the matrix of the Fourier-Mukai transform Φ_1 which is acting on the line $s = \frac{3}{2d}$, we have three different cases which are:

1. $\gcd(d,3) = 1$: Take $x = 3$, $b = d$, $a = x = 1$and $3z - 2dw = 1$. The radius of the walls are bounded by

$$\frac{1}{6d} \leq R \leq \frac{3}{2d}. \tag{6.4.6}$$

   This case is separated as the following

   (a) If $d \equiv 1 \mod 3$, then $w = 1$, $z = \dfrac{1+2d}{3}$, and the matrix is given by

$$\Phi = \begin{pmatrix} 9 & -6\ell & 4d \\ -3 & 1+4d & -\frac{2+4d}{3}\ell \\ d & -\dfrac{1+2d}{3}\ell & (\dfrac{1+2d}{3})^2 \end{pmatrix} \tag{6.4.7}$$

   and $\mathrm{ch}(\Phi_1((0,\ell,3))) = (0,\ell,\dfrac{1+2d}{3}) = (0,\ell,z)$.

(b) If $d \equiv 2 \mod 3$, then $z = \dfrac{1+4d}{3}$, $w = 2$ and the matrix is given by

$$\Phi = \begin{pmatrix} 9 & -6\ell & 4d \\ -6 & 1+6d & -\frac{2+8d}{3}\ell \\ 4d & -\frac{2+8d}{3}\ell & (\dfrac{1+4d}{3})^2 \end{pmatrix} \tag{6.4.8}$$

and $\mathrm{ch}(\Phi_1((0,\ell,3))) = (0,-\ell,\dfrac{1+4d}{3}) = (0,-\ell,z)$

2. $\gcd(d,3) = 3$: Take $b = \frac{d}{3}$, $a = 3$, $x = 1$ and $3z - \frac{2d}{3}w = 1$. The radius of the walls are bounded by

$$\frac{1}{2d} \le R \le \frac{3}{2d}. \tag{6.4.9}$$

This case is separated as the following

(a) If $\frac{d}{3} \equiv 1 \mod 3$, then $z = \frac{3+2d}{9}$, $w = 1$ and the matrix is given by

$$\Phi = \begin{pmatrix} 3 & -2\ell & \frac{d}{3} \\ -1 & \frac{3+4d}{3} & -\frac{6+4d}{9}\ell \\ \frac{d}{3} & -\frac{3+2d}{9}\ell & \frac{(3+2d)^2}{27} \end{pmatrix} \tag{6.4.10}$$

and $\mathrm{ch}(\Phi_1((0,\ell,3))) = (0,-\ell,\frac{3+2d}{3}) = (0,-\ell,az)$. The radius of the walls are bounded by

$$\frac{1}{6d} \le R \le \frac{3}{2d}. \tag{6.4.11}$$

(b) If $\frac{d}{3} \equiv 2 \mod 3$, then $z = \frac{3+4d}{9}$, $w = 2$ and the matrix is given by

$$\Phi = \begin{pmatrix} 3 & -2\ell & \frac{4d}{3} \\ -2 & \frac{3+8d}{3} & -\frac{6+8d}{9}\ell \\ \frac{4d}{3} & -\frac{6+8d}{9}\ell & \frac{(3+4d)^2}{27} \end{pmatrix} \tag{6.4.12}$$

and $\mathrm{ch}(\Phi_1((0,\ell,3))) = (0,-\ell,\frac{3+4d}{3}) = (0,-\ell,az)$.

6.4.5 $\beta = \lceil \frac{d}{2} \rceil$

If d is even and $4 \nmid d$, then fix $s = \frac{1}{4}$ and the condition is given by

$$\chi - \frac{d}{2} + \frac{r}{4} > 0. \tag{6.4.13}$$

The centre of the biggest wall is $C_0 = \frac{1}{4}$. Since $A \in T_{\frac{1}{4}}$, then $\mu(A) = \frac{2d}{r} > \frac{d}{2}$ and so $r < 4$. To get the matrix of the Fourier-Mukai transform Φ_1 which is acting on the line $s = \frac{1}{4}$, take $a = \frac{d}{2}$, $x = z = 1$, $b = 2$ and $w = \frac{d-2}{4}$ and we get

$$\Phi = \begin{pmatrix} \frac{d}{2} & -2\ell & 8 \\ -\frac{d-2}{4}\ell & d-1 & -2\ell \\ \frac{(d-2)^2}{8} & -\frac{d-2}{4}\ell & \frac{d}{2} \end{pmatrix} \tag{6.4.14}$$

The radius of the walls are bounded by

$$\frac{1}{2d} \leq R \leq \frac{1}{4}. \tag{6.4.15}$$

6.4.6 $\beta = d$

As it shown before that every destabilizer A of T has $c_1(A) = 1$. Fix $s = \frac{1}{2}$, the condition is given by

$$4\chi + rd > 4d. \tag{6.4.16}$$

Proposition 6.4.4. *If T is a torsion with $\mathrm{ch}(T) = (0, \ell, d)$, then $W(r, d, 1) = \left\lceil \frac{d}{4} \right\rceil$.*

Proof. The centre of the biggest wall is $C_0 = \frac{1}{2}$ and the radius is

$$R_0 = \sqrt{\frac{4(\chi - d) + rd}{4dr}}.$$

Applying the Bogomolov's inequality on A, the rank of A is bounded above by 2 and from semistability of $H^{-1}(B)$ we have $\chi \leq d$. Therefore at $\beta = d$ the rank of destabilizers is fixed and χ varies between $\frac{3d}{4} < \chi \leq d$ and the number of walls is $W(r, d, 1) = \left\lceil \frac{d}{4} \right\rceil$. □

Lemma 6.4.5. *For each β such that* $\gcd(\beta, d) \parallel d$*, the Fourier-Mukai transformation* $\Delta\Phi_1$ *preserves the Chern character of* $\mathrm{ch} = (0, \ell, \beta)$.

Proof. To get the matrix of a Fourier-Mukai transformation Φ_1 when $\beta = d$, we have two cases

i If d is odd, to get the matrix of Fourier-Mukai transformation Φ_1 who is acting on the line $s = 1$, take $a = d$, $x = w = 1$ and $w = \frac{d-1}{2}$ in (6.3.1). Then the Fourier-Mukai transformation is given by

$$\mathrm{ch}(\Delta\Phi_1(\mathrm{ch}(0, \ell, \beta))) = \begin{pmatrix} d & -2\ell & 4 \\ \frac{1-d}{2} & 2d-1 & -2\ell \\ \frac{(d-1)^2}{4} & \frac{1-d}{2}\ell & d \end{pmatrix} \begin{pmatrix} 0 \\ \ell \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ \ell \\ d \end{pmatrix}.$$

ii If d is even, take $a = \frac{d}{2}$, $z = 2$, $x = 1$ and $w = \frac{d-1}{2}$ in (6.3.2). Then we get

$$\mathrm{ch}(\Delta\Phi_1(\mathrm{ch}(0, \ell, \beta))) = \begin{pmatrix} 2d & -\ell & 2 \\ 1-d & 2d-1 & -2\ell \\ 2(d-1)^2 & (1-d)\ell & d \end{pmatrix} \begin{pmatrix} 0 \\ \ell \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ -\ell \\ d \end{pmatrix}.$$

□

Proposition 6.4.6. *When β is odd, the minimum wall is given by the destabilizer A with* $\mathrm{ch}(A) = (d, \frac{1-d}{2}\ell, (d-1)^2/4)$.

Proof. Suppose that the minimum wall is given by $\mathrm{ch} = (\hat{r}, \hat{c}\ell, \hat{\chi})$. Then

$$R^2_{\min} = \frac{\hat{\chi} - \hat{c}d}{d\hat{r}} + \frac{1}{4}.$$

From Remark 6.3.5 we have $R^2_{\min} = \frac{1}{4d^2}$ and so

$$\hat{\chi} = \hat{r}\left(\frac{1-d}{d}\right) + \hat{c}d \tag{6.4.17}$$

Since $r|d$, then take $r = \alpha d$. Therefore, $\hat{c} = \frac{\alpha}{2}d + 1$ and $\hat{\chi} = \alpha d^2 + d + 1$. Hence $\alpha = 1$ and $\mathrm{ch}(A) = (d, \frac{1-d}{2}\ell, (d-1)^2/4)$. □

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